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SINGULARITIES OF THE ANALYTIC TORSION

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1. Introduction

A construction, invented by D.B. Ray and I.M. Singer [23], [24], uses zeta-functions of Laplacians to assign to an elliptic complex (equipped with an inner product) a positive real number ρ, called *analytic torsion*. Ray and Singer themselves considered the analytic torsion for the De Rham and Dolbeaut complexes. It was A.S.Schwarz [26] who first studied the analytic torsion for general elliptic complexes. It is clear that the theory of analytic torsion in this generality has potentially a very wide field of possible applications in algebraic geometry, complex analysis and in mathematical physics.

A remarkable theorem, which was conjectured by Ray and Singer [23] and then proven later by J.Cheeger [6] and W.Müller [19], states that in the case of a De Rham complex, twisted by an orthogonal representation, the analytic torsion coincides with the classical Reidemeister-Franz-De Rham torsion, constructed using "finite"information on the manifold (namely, its cell decomposition). A more general theorem relating analytic torsion of the De Rham complex to the R-torsion was found recently by J.-M.Bismut and W.Zhang [5].

Suppose now, that the original elliptic complex is being deformed; this means that the differential operators, forming the complex, vary with a parameter t, where $t \in (a, b)$. Then the analytic torsion $\rho(t)$ will be a function of the parameter t. Even if the deformation of the differentials is analytic, the torsion $\rho(t)$ will in general have singularities (zeros and poles). The nature of these singularities is related to changes in the cohomology. In fact, the beautiful geometrical picture of the analytic torsion, suggested by D.Quillen [22] (cf. also [3]), con-

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sists in understanding it as a factor, which *smooths* the metric on the determinant line bundle coming from the Hodge decomposition. This shows that the problem of describing the singularities of the torsion is in some sense equivalent to the problem of describing the changes in cohomology which occur in the course of the deformation.

To study deformations of elliptic complexes we use in this paper the *germ-complex* of the deformation - a construction, which incorporates infinitesimal information about the deformation and allows one to view a single complex instead of a family of complexes. A similar idea was first exploited in [8] in order to describe in homological terms the jumps of the eta-invariant. It is shown in the present paper, that the cohomology of the germ-complex is finitely generated, if understood as a module over the ring, $\mathcal{O}$, of germs of holomorphic curves in $\mathbb{C}$ (with respect to pointwise addition and multiplication). The ring $\mathcal{O}$ is a principal ideal domain, and thus the cohomology of the germ complex can be decomposed into its free and torsion parts. In Theorem 2.8 it is shown that the rank of the free part of the germ-cohomology is equal to the dimension of the cohomology at generic points, and a precise relation is found between the cohomological jump and the torsion in the germ-cohomology.

As a consequence of this theorem we obtain Morse inequalities for deformations of elliptic complexes. These inequalities generalize Theorem 4.13 of [27] on upper semicontinuouty of dimensions of kernels of elliptic operators. Note also, that deformations of complex structures correspond to deformations of the Dolbeaut complex of a particular form; knowing this, one may easily deduce some theorems of T. Kodaira [14], Theorem 4.4, Theorem 7.8, and Theorem 7.13 from our Morse inequalities.

An important technical result, which is proven in the paper, is a Hodge decomposition theorem for the germ-complex; the proof of this theorem is based on the Rellich-Kato theorem on perturbations of self-adjoint operators.

Another interesting object which can be associated with a deformation of an elliptic complex is the spectral sequence of the deformation; cf. §6. In terms of this spectral sequence, one defines two numbers: the θ – torsion of the spectral sequence, and the χ – Euler number of the deformation, cf.§2. The latter can be expressed as a sum of jumps of the derived Euler characteristic (introduced by Bismut and Zhang [5]) of the terms of the spectral sequence. The following statement, describing the singularity of the analytic torsion as a function of a parameter, is the main result of the paper.

Theorem. *Let* $(C^\infty(\mathcal{E}), d_t)$ *be a deformation of an elliptic complex (cf. §2 for the definition) defined for* $t \in (a,b)$. *Then for any* $t_0 \in (a,b)$ *there exist* $\delta > 0$ *and a real analytic function* $f(t)$ *defined for* $t \in (t_0 - \delta, t_0 + \delta)$ *such that for all* $t \in (t_0 - \delta, t_0 + \delta)$, $t \neq t_0$, *the analytic torsion* $\rho(t)$ *is given by the formula*

$$\rho(t) \ = \ \theta \cdot |t - t_0|^{-\chi} \rho(t_0) \exp((t - t_0) f(t)),$$

where χ *is the Euler number of the deformation and* θ *is the torsion of the spectral sequence, associated with the deformation; cf. §6.*

This theorem follows immediately from Theorems 5.3 and 6.6 in the text of the paper.

As an application, we consider in §7 deformatons of flat vector bundles. It was shown in [8], that in this case, the germ-cohomology can be computed as usual cohomology of a local system determined by the deformation of the monodromy representation. We also observe some duality relations which exist in this situation because of the existence of the Hodge star operator.

In the last section §8 we consider a few other examples. First we point out some interesting relations between the point of view of the present work and the Alexander modules, which are among the most important tools of the knot theory. Using this connection we construct a curious example of a flat vector bundle which has no semi-simple deformations. We also discuss a recent result of A.Dimca and M.Saito on deformations of the Koszul complex, which exhibits entirely different properties.

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2. Deformations of elliptic complexes and their germ-cohomology

In this section we will study families of elliptic complexes obtained by deforming the differentials of a given elliptic complex. The first question which arises is about variation of the homology: it becomes to be a function of the parameter whose values are finite-dimensional vector spaces; the dimension of these vector spaces jumps at certain values of the parameter. To study those jumps we consider the deformation as a single complex of modules over the ring $\mathcal{O}$ of germs of holomorphic curves; we show that the cohomology of this complex (which we call *germ-complex*) determines the behaviour of the cohomology of the elliptic complex for all values of the parameter in a neibourhood of the fixed value. As a consequence we obtain Morse inequalities for deformations of elliptic complexes.

We will see later that information contained in the germ-complex is also useful in describing the analytic torsion as a function of the parameter.

2.1. Consider a closed smooth manifold M, complex vector bundles $\mathcal{E}_i$ over M, $0 \le i \le N$, and a set of first order linear differential operators,

$$d^i : C^\infty(\mathcal{E}_i) \to C^\infty(\mathcal{E}_{i+1}), \qquad 0 \le i < N,$$

(where $C^\infty(\mathcal{E})$ denotes the space of smooth sections of $\mathcal{E}$) which form an *elliptic complex.* This means that

(1) the sequence

$$0 \to C^\infty(\mathcal{E}_0) \xrightarrow{d^0} C^\infty(\mathcal{E}_1) \xrightarrow{d^1} \cdots \to C^\infty(\mathcal{E}_N) \to 0$$

is a complex, i.e., $d^{i+1} \circ d^i = 0$; and

(2) the associated symbol sequence

$$0 \longrightarrow \pi^*\mathcal{E}_0 \xrightarrow{\sigma(d^0)} \pi^*\mathcal{E}_1 \xrightarrow{\sigma(d^1)} \dots \xrightarrow{\sigma(d^{N-1})} \pi^*\mathcal{E}_N \longrightarrow 0$$

is exact in each fiber; here $\pi : S^*M \to M$ denotes the natural projection of the unit sphere subbundle of the cotangent bundle of M.

An elliptic complex as above will be denoted $(C^\infty(\mathcal{E}), d)$.

With an elliptic complex one associates its *cohomology*

$$H^i(C^\infty(\mathcal{E}), d) = \ker\,(d^{i+1}) / \operatorname{im}\,(d^i),$$

which is (as it is well-known) a finite dimensional vector space over $\mathbb{C}$.

2.2. Deformations of elliptic complexes. By a *deformation* of an elliptic complex $(C^\infty(\mathcal{E}), d)$ we will understand an object consisting of a family of elliptic complexes $(C^\infty(\mathcal{E}), d_t)$ given on the same set of vector bundles, where the differential operators

$$d_t^i : C^\infty(\mathcal{E}_i) \to C^\infty(\mathcal{E}_{i+1})$$

depend on a parameter t varying within an interval $(-\epsilon, \epsilon)$ such that:

(1) $(C^\infty(\mathcal{E}), d_t)$ is an elliptic complex for any $t \in (-\epsilon, \epsilon)$;

(2) for $t = 0$ we obtain the original unperturbed complex $(C^\infty(\mathcal{E}), d)$;

(3) for any $0 \leq i \leq N$ the first order differential operator $d_t^i \in \mathrm{Diff}_1(\mathcal{E}_i, \mathcal{E}_{i+1})$ depends *analytically* on the real parameter $t \in (-\epsilon, \epsilon)$. In other words, we suppose that the curve $(-\epsilon, \epsilon) \to \mathrm{Diff}_1(\mathcal{E}_i, \mathcal{E}_{i+1})$, given by d_t^i, is analytic. The precise meaning of analyticity of a family of operators will be described below in subsection 2.4. Intuitively, a curve of differential operators is analytic if all coefficients are real analytic functions of the parameter.

As an example consider the De Rham complex $(\Lambda^*(M), d)$ of the manifold M. Let ω be a fixed closed 1-form on M. S. P. Novikov [20] studied the following deformation

$$d_t = d \; + \; t\omega \wedge \cdot \quad , \qquad t \in \mathbb{R}.$$

In the case where the form ω is exact, $\omega = dh$, h being a Morse function on M, this deformation was studied by E.Witten [28].

Deformations of the Dolbeaut complex, restricted to lie in different subalgebras of the algebra of differential operators, correspond to a number of different deformation problems in complex analysis and algebraic geometry; cf. [14], [1], [22], [13].

2.3. The aim of this and of the following subsection is to make precise the notion of analyticity used in 2.2 in the definition of deformation of elliptic complex.

First we recall some standard definitions. Let Ω be an open subset of $\mathbb{C}$ and let V be a complex topological vector space. A function $f : \Omega \to V$ is said to be *weakly holomorphic in* Ω if vf is holomorphic in the ordinary sense for every continuous linear functional v on V. The function $f : \Omega \to V$ is said to be *strongly holomorphic in* Ω if the limit

$$\lim_{w\to z}\frac{f(w)-f(z)}{w-z}$$

exists (in the topology of V) for every $z \in \Omega$. It is known that the above two notions of analyticity actually coincide if V is a Frechet space; cf. [25 (Chapter 3)].

A function $f : (a,b) \to V$ defined on a real interval (a,b) with values in a Frechet space V is said to be *analytic (or real analytic or holomorphic)* if it is a restriction of an analytic function $\Omega \to V$ defined in a neighbourhood $\Omega \subset \mathbb{C}$ of the interval (a,b).

We will mainly consider analytic curves in spaces of smooth sections of vector bundles. Let M be a compact C^∞ Riemannian manifold (possibly with boundary) and let $\mathcal{E}$ be a Hermitian vector bundle over M. For any integer k symbol $\mathcal{H}_k(\mathcal{E})$ will denote the corresponding Sobolev space (defined as in Chapter 9 of [21]). Recall that the Sobolev spaces $\mathcal{H}_k(\mathcal{E})$ with $k \in \mathbb{Z}$ form a chain of Hilbertian spaces (in the terminology of [21]), which, in particular, means that $\mathcal{H}_k(\mathcal{E})$ is embedded into $\mathcal{H}_l(\mathcal{E})$ for $k > l$ (as a topological vector space) and the intersection of all the spaces $\mathcal{H}_k(\mathcal{E})$ coincides with $\mathcal{H}_\infty(\mathcal{E}) = C^\infty(M)$.

Definition. Let $f : (a,b) \to C^\infty(M)$ be a curve of smooth sections; we will say that f is *analytic* if for any integer k the curve f represents a (real) analytic curve considered as a curve in the Sobolev space $\mathcal{H}_k(\mathcal{E})$. In other words, for any k there is a neighbourhood $U_k \subset \mathbb{C}$ of the interval (a,b) and an analytic function $f_k : U_k \to \mathcal{H}_k(\mathcal{E})$ extending f.

Note that any curve $f : (a,b) \to C^\infty(M)$ which is analytic by viewing $C^\infty(M)$ as a Frechet space, will be obviously analytic in the sense of above Definition. The converse is also true, although the proof of this fact (shown to me by V.Matsaev) is not elementary; it uses interpolation theory of Hilbert spaces. Since we wish to avoid these analytic subtleties, and since the above definition is the most convenient and entirely sufficient for our purposes, we will accept it and never use the equivalence of the above two definitions in the present paper.

Suppose that $\mathcal{E}$ and $\mathcal{F}$ are two Hermitian vector bundles over M. Then any differential operator $D : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{F})$ of order ℓ defines a bounded linear map of Sobolev spaces $\mathcal{H}_k(\mathcal{E}) \to \mathcal{H}_{k-\ell}(\mathcal{F})$ (where $k \geq \ell$) and thus D maps analytic curves in $C^\infty(\mathcal{E})$ into analytic curves in $C^\infty(\mathcal{F})$.

Let $\mathcal{O}$ denote the ring of germs of analytic curves in $\mathbb{C}$. Addition and multiplication are given by pointwise operations. $\mathcal{O}$ is a discrete valuation ring; its maximal ideal $\mathfrak{m} = (t) \subset \mathcal{O}$ coincides with the set of all functions vanishing at the origin. The generator of the maximal ideal is the germ of the function $f(t) = t$.

For a vector bundle $\mathcal{E}$ let $\mathcal{O}C^\infty(\mathcal{E})$ denote the set of germs of analytic curves in $C^\infty(\mathcal{E})$ in the sense described above. Then the pointwise product of curves makes $\mathcal{O}C^\infty(\mathcal{E})$ to be an $\mathcal{O}$-module.

2.4. We will give now definition of analyticity for families of linear differential operators.

Suppose that $\mathcal{E}$ and $\mathcal{F}$ are two vector bundles over the manifold M, and $D_t \in \mathrm{Diff}_\ell(\mathcal{E}, \mathcal{F})$ is a family of linear differential operators of order ℓ, depending on a real parameter $t \in (a, b)$. Let $J^\ell(\mathcal{E})$ denote the jet bundle of order ℓ; cf. [21 (chapter IV, §2)]. Then by Theorem 1 on page 61 of [21], the set $\mathrm{Diff}_\ell(\mathcal{E}, \mathcal{F})$ can be identified with $C^\infty(\mathrm{Hom}(J^\ell(\mathcal{E}), \mathcal{F}))$. The latter is the set of smooth sections of a vector bundle; therefore we can consider analytic curves in this space of sections using the definition of analyticity given in 2.3.

We accept the following definition: a curve of linear differential operators $(a, b) \to \mathrm{Diff}_\ell(\mathcal{E}, \mathcal{F})$ is said to be *(real) analytic* iff the corresponding curve of sections of the bundle $\mathrm{Hom}(J^\ell(\mathcal{E}), \mathcal{F})$ is analytic.

The main property of analytic families of operators D_t, which we will constantly use, consists of the following: for any integer $k \geq \ell$ the family of bounded linear operators $D_t : \mathcal{H}_k(\mathcal{E}) \to \mathcal{H}_{k-\ell}(\mathcal{F})$ depends analytically on the parameter t (i.e., defines an analytic curve in the Banach space of bounded linear operators $\mathcal{H}_k(\mathcal{E}) \to \mathcal{H}_{k-\ell}(\mathcal{F})$ with the operator norm).

From the above remark it follows that *if $f : (a, b) \to C^\infty(\mathcal{E})$ is an analytic curve of smooth sections, and $D : (a, b) \to \mathrm{Diff}_\ell(\mathcal{E}, \mathcal{F})$ is an analytic curve of linear differential operators, then the "evaluation curve" $t \mapsto D_t(f_t)$ is also analytic.*

2.5. The germ-complex. Suppose again that we are given an elliptic complex

$$0 \to C^\infty(\mathcal{E}_0) \xrightarrow{d^0} C^\infty(\mathcal{E}_1) \xrightarrow{d^1} \dots C^\infty(\mathcal{E}_N) \to 0$$

over M. A deformation of this complex (defined as in 1.2) determines a one-parameter family of chain complexes of global sections

$$0 \to C^\infty(\mathcal{E}_0) \xrightarrow{d_t^0} C^\infty(\mathcal{E}_1) \xrightarrow{d_t^1} \dots C^\infty(\mathcal{E}_N) \to 0$$

which can be understood as the following single complex of $\mathcal{O}$-modules and $\mathcal{O}$-homomorphisms

$$(1) \qquad 0 \to \mathcal{O}C^\infty(\mathcal{E}_0) \xrightarrow{\tilde{d}} \mathcal{O}C^\infty(\mathcal{E}_1) \xrightarrow{\tilde{d}} \dots \mathcal{O}C^\infty(\mathcal{E}_N) \to 0;$$

the differential $\tilde{d} : \mathcal{O}C^\infty(\mathcal{E}_i) \to \mathcal{O}C^\infty(\mathcal{E}_{i+1})$ is given by

$$\tilde{d}(s)(t) = d_t^i(s(t)).$$

where $s : (-\epsilon, \epsilon) \to C^\infty(\mathcal{E}_i)$ is a curve of sections.

We will call (1) *the germ-complex of the deformation.*

Cohomology of complex (1) in dimension i, denoted $H^i(\mathcal{O}C^\infty(\mathcal{E}), \tilde{d})$, will be called the *germ-cohomolody or $\mathcal{O}$-cohomology of the deformation.* Note that the $\mathcal{O}$-cohomology is an $\mathcal{O}$-module.

2.6. Finiteness Theorem. *The germ-cohomology modules of a deformation of an elliptic complex over a compact manifold are finitely generated as modules over $\mathcal{O}$.*

The proof will be given later in section 3.3.

It was observed above, that $\mathcal{O}$ is a discrete valuation ring. Thus, any finitely generated module X over $\mathcal{O}$ can be represented as a direct sum $F \oplus \tau$ of a finitely generated *free* module F and the *torsion* submodule $\tau \subset X$. *Rank* of X is defined as the rank of F. Any finitely generated torsion $\mathcal{O}$-module τ is a direct sum of *cyclic modules* (i.e., modules of the form $\mathcal{O}/t^n\mathcal{O}$, where $n \in \mathbb{N}$) and this direct sum representation is unique. We will denote by $\mu(\tau)$ the number of the cyclic summands contained in the decomposition of τ.

2.7. Definition. Let τ^i denote the torsion submodule of $H^i(\mathcal{O}C^\infty(\mathcal{E}), \tilde{d})$, the germ-cohomology module of the given deformation. The Euler number of the deformation is defined as

$$(2) \qquad \chi = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{C}} \tau^i.$$

From the finiteness theorem it follows that τ^i is finitely dimensional as a vector space and so the above definition makes sense.

We will see later in §5 that this number χ *determines the singularity of the analytic torsion as function of the parameter.*

Remark. It is clear that when a deformation is defined for all values of a parameter varying within an interval, any point of this interval can be fixed and the complex of curves' germs at this particular point can be studied (instead of the origin $t = 0$ as above). In this case the ring $\mathcal{O}$ must be substituted by the ring of germs of holomorphic curves at that point. Thus the notions of the germ-cohomology and the Euler number χ of the deformation introduced above are *relative* and can be defined with respect to any point of the interval of the values of the parameter.

Our immediate task now is to show that the germ-cohomology of the deformation determines the cohomology of all elliptic complexes $(C^\infty(\mathcal{E}), d_t)$ for all values of t close to 0.

Theorem 2.8. *Suppose that $(C^\infty(\mathcal{E}), d_t)$ is a deformation of an elliptic complex over a closed manifold M, where $t \in (-\epsilon, \epsilon)$. Then*

(a) *there exists a positive $\delta < \epsilon$ such that for all values of the parameter t satisfying $0 < |t| < \delta$ the dimension of the cohomology of the elliptic complex $(C^\infty(\mathcal{E}), d_t)$ (where t is fixed) is equal to the rank of the $\mathcal{O}$-cohomology module:*

$$\dim_{\mathbb{C}} H^i(C^\infty(\mathcal{E}), d_t) = \mathrm{rk}_{\mathcal{O}}\, H^i(\mathcal{O}C^\infty(\mathcal{E}), \tilde{d}).$$

(b) *the dimension of the cohomology of the undeformed complex*

$$\dim H^i(C^\infty(\mathcal{E}), d_0)$$

is equal to

$$\mathrm{rk}_{\mathcal{O}}\, H^i(\mathcal{O}C^\infty(\mathcal{E}), \tilde{d}) + \mu(\tau^i) \ + \mu(\tau^{i+1}),$$

where τ^i denotes the torsion submodule of $H^i(\mathcal{O}C^\infty(\mathcal{E}), \tilde{d})$.

Note that the statement (1) of the Theorem 2.8 describes the cohomology of the elliptic complex $(C^\infty(\mathcal{E}), d_t)$ for *generic* t.

The proof of Theorem 2.8 is given in section 4.

We will discuss in section 8 some examples of deformations of Koszul complexes which exhibit entirely different behaviour.

As an immediate consequence of Theorem 2.8 we obtain Morse inequalities for deformations of elliptic complexes:

2.9. Theorem (Morse inequalities). *Assume that a deformation of an elliptic complex $(C^\infty(\mathcal{E}), d_t)$ over a compact manifold is given. Let $b_i(t)$ denote the Betti numbers $\dim H^i(C^\infty(\mathcal{E}), d_t)$ of the elliptic complex as functions of the parameter t varying in $(-\epsilon, \epsilon)$. Then there exists δ with $0 < \delta < \epsilon$ such that $b_i(t)$ assumes the same constant value for all*

$t \in (-\delta, \delta)$, $t \neq 0$, *and for those* t *the following inequalities hold*

$$\sum_{j=0}^{i}(-1)^j b_{i-j}(t) \leq \sum_{j=0}^{i}(-1)^j b_{i-j}(0) \qquad i = 0, 1, 2, \ldots.$$

This theorem generalizes Theorem 4.13 of [27] on upper semicontinuouty of the kernels of elliptic operators. Some theorems of K.Kodaira [14], Theorems 4.4, 7.8, 7.13, follow from Theorem 2.9.

2.10. Following Bismut and Zhang [5] we will associate to an elliptic complex $(C^\infty(\mathcal{E}), d)$ the number

$$\chi' = \sum_{i=0}^{\infty}(-1)^i i \dim H^i(C^\infty(\mathcal{E}), d),$$

called *derived Euler characteristics.*

Given a deformation $(C^\infty(\mathcal{E}), d_t)$, this number $\chi' = \chi'(t)$ will depend on the parameter t, $-\epsilon < t < \epsilon$. It will have a jump at $t = 0$, and Theorem 2.8 gives the value of the jump:

$$\chi'(0) - \chi'(t) = \sum_{i=0}^{\infty}(-1)^i \mu(\tau^i),$$

for all $-\delta < t < \delta$, $t \neq 0$.

2.11. Definition. A deformation of an elliptic complex $(C^\infty(\mathcal{E}), d_t)$, where $-\epsilon < t < \epsilon$, is said to be *semi-simple* if all torsion homology modules τ^i of the germ-complex are semi-simple; the last condition is equivalent to either of the conditions:

$$\mu(\tau^i) = \dim_{\mathbb{C}}(\tau^i) \qquad \text{and} \qquad t \cdot \tau^i = 0$$

for all i. From Theorem 2.8 and the above remark we obtain:

2.12. Corollary. *In the semi-simple case, the Euler number* χ *of deformation equals to the jump of the derived Euler characteristics. In other words,*

$$\chi = \chi'(0) - \chi'(t)$$

for nonzero t *close to* 0.

I want to emphasize that the previous formula does not hold for deformations which are not semi-simple, as easy examples show. A general formula for the number χ will be described in 6.3; it involves the derived Euler characteristics of a spectral sequence associated with the deformation of the elliptic complex.

One may suspect that semi-simple deformations must be generic. But surprisingly it is not true: in §8 we will describe an example in which *all deformations* (of a flat vector bundle) are not semi-simple.

3. Parametrized Hodge decomposition

In this section we study the decomposition of the space of germs of curves in an elliptic complex which is quite analogous to the Hodge decomposition. It has however one main difference from the classical situation which consists in the necessity to use the periodic closure $\mathfrak{Cl}$; the reason for this is the fact that the set of germs of analytic curves $\mathcal{O}$ is a ring but not a field. The arguments of the proof are based on the theory of perturbations of self-adjoint operators, described in the book of T. Kato [12]. We conclude this section by a proof of the finiteness theorem 2.6.

3.1. Consider a deformation $(C^\infty(\mathcal{E}), d_t)$ of an elliptic complex over a closed manifold M, where $t \in (-\epsilon, \epsilon)$, and the corresponding germ-complex (1)

$$0 \to \mathcal{O}C^\infty(\mathcal{E}_0) \xrightarrow{\tilde{d}} \mathcal{O}C^\infty(\mathcal{E}_1) \xrightarrow{\tilde{d}} \cdots \to \mathcal{O}C^\infty(\mathcal{E}_N) \to 0.$$

Suppose now that each vector bundle $\mathcal{E}_i$ is supplied with a Hermitian metric, and a Riemannian metric is fixed on M. Then there is a natural scalar product on the spaces of smooth sections $C^\infty(\mathcal{E}_i)$: for sections $s,\ s' \in C^\infty(\mathcal{E}_i)$ their scalar product is given by

$$(s, s') \;=\; \int_M (s_x, s'_x)dx.$$

Extending this scalar product pointwise on the space of of curves we obtain the Hermitian pairing of $\mathcal{O}$-modules

$$(3) \qquad (\, , \,): \quad \mathcal{O}C^\infty(\mathcal{E}_i) \otimes \mathcal{O}C^\infty(\mathcal{E}_i) \to \mathcal{O}.$$

Each operator $d_t^i : C^\infty(\mathcal{E}_i) \to C^\infty(\mathcal{E}_{i+1})$ has adjoint (with respect to the Hermitian scalar product mentioned above) which we will denote

$$\delta_t^i : C^\infty(\mathcal{E}_{i+1}) \to C^\infty(\mathcal{E}_i).$$

As above, the family of operators δ_t^i defines the operator $\tilde{\delta}$ acting on curves

$$\tilde{\delta} : \mathcal{O}C^\infty(\mathcal{E}_i) \to \mathcal{O}C^\infty(\mathcal{E}_{i+1}), \qquad \tilde{\delta}(s(t))(x) = \delta_t(s(t))(x),$$

where $s(t) \in C^\infty(\mathcal{E}_i)$ is a holomorphic curve of smooth sections defined for $-\epsilon < t < \epsilon$. We will also need the Laplacians

$$\tilde{\Delta} = \tilde{d}\tilde{\delta} + \tilde{\delta}\tilde{d} : \mathcal{O}C^\infty(\mathcal{E}_i) \to \mathcal{O}C^\infty(\mathcal{E}_i).$$

As usual, elements of the kernel of the Laplacian will be called harmonic forms; the set of all harmonic forms in $\mathcal{O}C^\infty(\mathcal{E}_i)$ will be denoted Har^i. Note that the operators $\tilde{\Delta}, \tilde{d}, \tilde{\delta}$ are $\mathcal{O}$-homomorphisms; in particular, we see that the space of harmonic forms is an $\mathcal{O}$-submodule.

Now we will introduce the algebraic notation which will be used later in the Hodge decomposition theorem for curves. For an $\mathcal{O}$-submodule X of an $\mathcal{O}$-module Y the symbol $\mathfrak{Cl}(X)$ will denote the pure submodule generated by X, i.e., the set of all elements $y \in Y$ with the property that fy belongs to X for some nonzero $f \in \mathcal{O}$.

The following is the main result of this section.

3.2. Theorem. *Suppose that a deformation $(C^\infty(\mathcal{E}), d_t)$ of an elliptic complex over a closed manifold M is defined for $t \in (-\epsilon, \epsilon)$. Then the following decomposition holds:*

$$\mathcal{O}C^\infty(\mathcal{E}_i) = \mathrm{Har}^i \oplus \mathfrak{Cl}(\tilde{d}(\mathcal{O}C^\infty(\mathcal{E}_{i-1})) \oplus \mathfrak{Cl}(\tilde{\delta}(\mathcal{O}C^\infty(\mathcal{E}_{i+1})),$$

and the terms of this decomposition are orthogonal to each other with respect to scalar product (3). *Moreover, the $\mathcal{O}$-module* Har^i *of harmonic forms is free of finite rank while the factormodules*

$$\tau^i = \mathfrak{Cl}(\tilde{d}(\mathcal{O}C^\infty(\mathcal{E}_{i-1})))/\tilde{d}(\mathcal{O}C^\infty(\mathcal{E}_{i-1}))$$

and

$$\varrho^i = \mathfrak{Cl}(\tilde{\delta}(\mathcal{O}C^\infty(\mathcal{E}_{i+1})))/\tilde{\delta}(\mathcal{O}C^\infty(\mathcal{E}_{i+1}))$$

are finitely generated torsion $\mathcal{O}$-modules.

We will need the following lemmas.

3.3. Lemma. *Let $\mathcal{E}$ be a Hermitian vector bundle over a compact Riemannian manifold M without boundary and let $D_t \in \mathrm{Diff}_\ell(\mathcal{E}, \mathcal{E})$ be an analytic (in the sense of 2.3) family of elliptic self-adjoint operators* *of order $\ell > 0$ defined for $t \in (a,b)$. Suppose that $\phi, \psi : (a,b) \to C^\infty(\mathcal{E})$ are two curves such that $D_t(\phi(t)) = \psi(t)$ for any $t \in (a,b)$, and it is known that the curve ψ is analytic in the sense of definition 2.3, while the curve ϕ is analytic in a weaker sense - as a curve in the Hilbert space $\mathcal{H}_0(\mathcal{E}) = L^2(\mathcal{E})$. Then the curve ϕ is analytic in the sense of definition 2.3 as well.*

Proof. Choose a point $t_0 \in (a,b)$ and an integer $k \geq 0$. It is enough to prove analyticity of the curve $\phi : (t_0 - \delta, t_0 + \delta) \to \mathcal{H}_k(\mathcal{E})$ for some small $\delta > 0$ (the restriction of the original curve ϕ onto a neighbourhood of t_0, considered as a curve in the Sobolev space $\mathcal{H}_k(\mathcal{E})$).

Let π denote the orthogonal projection of $\mathcal{H}_0(\mathcal{E})$ onto $\ker(D_{t_0}) \subset \mathcal{H}_\infty(\mathcal{E})$. The operator

$$D_t + \pi : \quad \mathcal{H}_k(\mathcal{E}) \to \mathcal{H}_{k-\ell}(\mathcal{E})$$

is continuous, analytically depends on the parameter t, and is invertible for $t = t_0$. Thus it is invertible for $t \in (t_0 - \delta, t_0 + \delta)$ for some $\delta > 0$. We have

$$(D_t + \pi)(\phi(t)) = \psi(t) + \pi(\phi(t)).$$

We claim that the right hand side of this equation is a curve analytic in the Sobolev space $\mathcal{H}_{k-\ell}(\mathcal{E})$. In fact, the first summand $\psi(t)$ is analytic in any Sobolev space by the assumption, while the second summand $\pi(\phi(t))$ belongs to a finite dimensional subspace $\ker D_{t_0}$, and it is given that it is analytic as a curve in Hilbert space $L^2(\mathcal{E}) = \mathcal{H}_0(\mathcal{E})$. Since all linear topologies on a finite dimensional vector space are equivalent, we conclude that the curve $\pi(\phi(t))$ is analytic as a curve in $\mathcal{H}_{k-\ell}(\mathcal{E})$.

Combining the remarks of the two previous paragraphs, we obtain that the curve $\phi : (t_0 - \delta, t_0 + \delta) \to \mathcal{H}_k(\mathcal{E})$ is analytic.

3.4. Lemma. *Let $\mathcal{E}$ be a Hermitian vector bundle over a compact Riemannian manifold M without boundary and let $D_t \in \mathrm{Diff}_\ell(\mathcal{E}, \mathcal{E})$ be an analytic (in the sense of 2.3) family of elliptic self-adjoint operators of order $\ell > 0$ defined for $t \in (a,b)$. Suppose that $\ker D_t = 0$ for all $t \in (a,b)$. If $\phi, \psi : (a,b) \to C^\infty(\mathcal{E})$ are two curves such that $D_t(\phi(t)) = \psi(t)$ for any $t \in (a,b)$ and the curve ψ is analytic (in the sense of definition 2.3), then the curve ϕ is also analytic.*

Proof. Fix an integer k. Since $\ker D_t = 0$, the operator D_t defines a linear homeomorphism $D_t : \mathcal{H}_{k+\ell}(\mathcal{E}) \to \mathcal{H}_k(\mathcal{E})$ (by the open mapping theorem, cf. [25 (p.47)]) which depends analytically on t. Thus it follows that $\phi(t) = D_t^{-1}(\psi(t))$ is an analytic curve in the Sobolev space $\mathcal{H}_{k+l}(\mathcal{E})$. Since this is true for any k, the statement follows.

3.5. Proof Theorem 3.2. First we want to show that there is an orthogonal decomposition

$$OC^\infty(\mathcal{E}_i) = \mathrm{Har}^i \oplus \mathfrak{Cl}(\tilde{\Delta}(OC^\infty(\mathcal{E}_i)). \tag{9}$$

Consider the family of Laplacians

$$\Delta_t : C^\infty(\mathcal{E}_i) \to C^\infty(\mathcal{E}_i), \quad (-\epsilon < t < \epsilon), \quad \text{where} \quad \Delta_t = d_t\delta_t + \delta_t d_t,$$

acting on the Hilbert space $\mathcal{H} = \mathcal{H}_0(\mathcal{E}) = L^2(\mathcal{E})$. It is a holomorphic family of self-adjoint elliptic operators in the sense of subsection 2.4; it is also a self-adjoint holomorphic family of type (A) in the terminology of T.Kato; cf. Chapter 7, §2 of [12]. The domain $\mathcal{D} \subset \mathcal{H}$ of Δ_t does not depent on t; it coincides with the Sobolev space $\mathcal{H}_2(\mathcal{E})$.

By Theorem 3.9 from [12], chapter 7, §3, we obtain the existence of *parametrized spectral decomposition.* The latter consists in a sequence of holomorphic curves $\phi_n(t) \in \mathcal{D} \subset \mathcal{H}$ (which are holomorphic with respect to the metric of $\mathcal{H} = \mathcal{H}_0(\mathcal{E})$), defined for $t \in (-\epsilon, \epsilon)$, for any $n \geq 1$, and a sequence of real valued holomorphic functions $\lambda_n(t)$, $n \geq 1$, $t \in (-\epsilon, \epsilon)$, such that for any value of t the numbers $\{\lambda_n(t)\}$ represent all the repeated eigenvalues of Δ_t, and $\{\phi_n(t)\}$ form a complete orthonormal family of associated eigenvectors of Δ_t. Observe that $\phi_n(t)$ are smooth, $\phi_n(t) \in C^\infty(\mathcal{E}_i) \subset \mathcal{D} \subset \mathcal{H}$, by the regularity theorem for elliptic operators.

Now Lemma 3.3 (applied to the operators $\Delta_t - \lambda_n(t)$) implies analyticity of the curves of eigenfunctions $\phi_n(t)$ in the strong $\mathcal{H}_\infty(\mathcal{E})$-sense.

From the general theory of elliptic operators we know that for any value of t the sequence of numbers $\{\lambda_n(t)\}$ tends to infinity. Thus there may exist only finitely many integers n with $\lambda_n(0) = 0$. Suppose that $\lambda_n(t)$ and $\phi_n(t)$ have been numerated in such a way that the following hold:

(i) $\lambda_n(t) \equiv 0$ for $1 \leq n \leq N_0$;

(ii) $\lambda_n(t) = t^{\nu_n}\overline{\lambda}_n(t)$ with $\nu_n \geq 1$, $\overline{\lambda}_n(0) \neq 0$ for $N_0 < i \leq N$;

(iii) $\lambda_n(0) \neq 0$ for $n > N$.

Any analytic curve $t \mapsto f_t, f \in OC^\infty(\mathcal{E}_i)$ can be represented in the form

$$f_t = \sum \beta_n(t)\phi_n(t), \qquad \beta_n \in \mathcal{O},$$

where the series converges in $\mathcal{H} = \mathcal{H}_0(\mathcal{E})$. Then

$$\Delta_t f_t = \sum \beta_n(t)\lambda_n(t)\phi_n(t).$$

Thus $\Delta_t f_t \equiv 0$ if and only if all coefficients β_n vanish for $n > N_0$. It means that the germs of curves

$$\phi_1(t),\ \phi_2(t), \dots, \phi_{N_0}(t)$$

belong to the space harmonic forms Har^i and furnish a free $\mathcal{O}$-basis of Har^i. This proves one of the statements of Theorem 3.2.

Suppose that a germ of a holomorphic curve $f : (-\epsilon, \epsilon) \to C^\infty(\mathcal{E}_i)$ belongs to the image of $\tilde{\Delta}$. Then obviously:

(a) $(f, \phi) = 0$ for any $\phi \in \mathrm{Har}^i$ and

(b) (f, ϕ_k) is divisible by t^{ν_k} in $\mathcal{O}$ for any $N_0 < k \leq N$,

where the scalar product (3) has been used.

Conversely, suppose that we are given an analytic curve $f : (-\epsilon, \epsilon) \to C^\infty(\mathcal{E}_i)$, $f \in \mathcal{O}C^\infty(\mathcal{E}_i)$, which satisfies the properties (a) and (b). Then in the decomposition $f_t = \sum \beta_n(t)\phi_n(t)$ we have: (1) $\beta_n(t) \equiv 0$ for $n \leq N_0$ and (2) $\beta_n(t)$ is divisible by $\lambda_n(t)$ in $\mathcal{O}$ for $N_0 < n \leq N$. Consider

$$g(t) = \sum_{n>N_0}^{N} \frac{\beta_n(t)}{\lambda_n(t)}\phi_n(t) + \sum_{n>N} \frac{\beta_n(t)}{\lambda_n(t)}\phi_n(t).$$

We claim that g is holomorphic as a curve in $\mathcal{H}_\infty(\mathcal{E}_i)$, i.e., $g \in \mathcal{O}C^\infty(\mathcal{E}_i)$. In fact,the first sum $g_1(t)$ is finite and it is holomorphic by condition (b). Considering the second sum (denoted by $g_2(t)$), note that $\lambda_n(0) > 0$ for $n > N$ and $\lambda_n(0) \to \infty$. From the analyticity of Δ_t it now follows that there exit positive numbers a and δ such that $\lambda_n(t) > a$ for all $n > N$ and $|t| < \delta$ (here one may use Theorem 2.5 of [11], for example). Thus, we see that the second term $g_2(t)$ converges as quickly as the series for f in any Sobolev space $\mathcal{H}_k(\mathcal{E}_i)$; thus g_2 is a curve with values in $C^\infty(\mathcal{E}_i)$, continuous in the topology $\mathcal{H}_\infty(\mathcal{E}_i)$.

Now we want to show that $g_2(t)$ is analytic in $\mathcal{H}_\infty(\mathcal{E}_i)$ for small t. For any value of the parameter t let π_t denote the orthogonal projection (with respect to the $L^2 = \mathcal{H}_0$-scalar product) onto the subspace generated by the eigenfunctions $\phi_n(t)$ with $n = 1, 2, \dots, N$. Denote

$L_t = \Delta_t + \pi_t$. We have $L_t(g_2(t)) = f(t)$ is a holomorphic curve in $\mathcal{H}_\infty(\mathcal{E}_i)$ and $\ker L_t = 0$. Thus we may apply Lemma 3.4 to conclude the analyticy of $g_2(t)$. Note, that L_t (being a sum of the Laplacian with a finite-dimensional projector, which depend on t analytically in the obvious sense) is a *pseudo-differential operator*; but all the arguments of the proof of Lemmas 3.3 and 3.4 still work.

Thus, we obtain that the orthogonal complement of Har^i in $\mathcal{O}C^\infty(\mathcal{E}_i)$ coincides with $\mathfrak{Cl}(\tilde{\Delta}(C^\infty(\mathcal{E}_i)))$ (which proves the decomposition (9)) and the factor-space $\mathfrak{Cl}(\tilde{\Delta}(C^\infty(\mathcal{E}_i)))/\tilde{\Delta}(C^\infty(\mathcal{E}_i))$ has finite dimension equal to

$$\sum_{k>N_0}^{N} \nu_k .$$

To complete the proof we need to show that

$$\mathfrak{Cl}(\tilde{\Delta}(C^\infty(\mathcal{E}_i))) = \mathfrak{Cl}(\tilde{d}(\mathcal{O}C^\infty(\mathcal{E}_{i-1}))) \oplus \mathfrak{Cl}(\tilde{\delta}(\mathcal{O}C^\infty(\mathcal{E}_{i+1}))). \tag{10}$$

If $f \in \mathfrak{Cl}(\tilde{\Delta}(C^\infty(\mathcal{E}_i)))$ then $t^l f = \tilde{\Delta} g$ for some positive integer l and $g \in (C^\infty(\mathcal{E}_i))$. The curves $h_1 = \tilde{d}\tilde{\delta} g$ and $h_2 = \tilde{\delta}\tilde{d} g$ are orthogonal to each other and

$$(t^l f, t^l f) = (h_1, h_1) + (h_2, h_2).$$

Since the right-hand-side is $O(t^{2l})$ we obtain that $h_j = t^l f_j, \quad j = 1, 2$ and $f_j \in \mathcal{O}C^\infty(\mathcal{E}_i)$. It follows that

$$f_1 \in \mathfrak{Cl}(\tilde{d}(\mathcal{O}C^\infty(\mathcal{E}_{i-1}))), \qquad f_2 \in \mathfrak{Cl}(\tilde{\delta}(\mathcal{O}C^\infty(\mathcal{E}_{i+1})))$$

and $f = f_1 + f_2$. This proves that right-hand-side of (10) is contained in its LHS.

Let us prove the inverse inclusion. If $f \in \mathfrak{Cl}(\tilde{d}(\mathcal{O}C^\infty(\mathcal{E}_{i-1})))$ then we can write $t^l f = \tilde{d} g$ for some $g \in \mathcal{O}C^\infty(\mathcal{E}_{i-1}))$. By (9) we may represent $g = h + g_1$ where h is harmonic and $t^s g_1 = \tilde{\Delta} g_2$. Then $t^{l+s} f = \tilde{\Delta}\tilde{d} g_2$ which proves that $f \in \mathfrak{Cl}(\tilde{\Delta}(\mathcal{O}C^\infty(\mathcal{E}_i)))$. Similar arguments applied to the second summand give the proof of (10).

Since $\mathrm{im}(\tilde{\Delta}) \subset \mathrm{im}(\tilde{d}) \oplus \mathrm{im}(\tilde{\delta})$, it follows from (10) that there is an epimorphism

$$\mathfrak{Cl}(\tilde{\Delta}(\mathcal{O}C^\infty(\mathcal{E}_i)))/\tilde{\Delta}(\mathcal{O}C^\infty(\mathcal{E}_i)) \to \tau^i \oplus \varrho^i$$

and thus τ^i and ϱ^i are finitely dimensional.

3.6. Proof of the Finiteness Theorem 2.6. Consider the action of the differential $\tilde{d}$ of the chain complex (1) on the terms of the orthogonal decomposition of $\mathcal{O}C^\infty(\mathcal{E}_i)$ given by Theorem 3.2. It obviously vanishes on the first two terms

$$\text{Har}^i \oplus \mathfrak{Cl}(\tilde{d}(\mathcal{O}C^\infty(\mathcal{E}_{i-1})), \tag{11}$$

and it is monomorphic on $\mathfrak{Cl}(\tilde{\delta}(\mathcal{O}C^\infty(\mathcal{E}_{i+1})),$ since from $t^l f = \tilde{\delta} g$ and $\tilde{d} f = 0$ it follows that $\tilde{d}\tilde{\delta} g = 0$ and

$$0 = (\tilde{d}\tilde{\delta} g, g) = (\tilde{\delta} g, \tilde{\delta} g);$$

thus $\tilde{\delta} g = 0$ and $f = 0$.

We obtain that (11) comprise the set of cycles of complex (1). Since the set of boundaries is obviously $\tilde{d}(\mathcal{O}C^\infty(\mathcal{E}_{i-1}))$ we find that the homology of (1) is $\text{Har}^i \oplus \tau^i$. It is finitely generated over $\mathcal{O}$ by Theorem 3.2.

4. Proof of Theorem 2.8

4.1. As it was shown in the proof of Theorem 3.2, the $\mathcal{O}$-module $\mathfrak{Cl}(\text{im}(\tilde{\Delta}))/\text{im}(\tilde{\Delta})$ has a very clear description in terms of the parametrized spectral decomposition of the Laplacians, given by the Theorem of T. Kato on perturbations of self-adjoint operators. Our aim now is to find the relations between this module and the modules τ^i and ϱ^i. These relations will allow us to study the behaviour of the cohomology and the analytic torsion as functions of the parameter.

To simplify the notation, we will denote in this section $C^i = \mathcal{O}C^\infty(\mathcal{E}_i)$ and will drop the tilde sign from the notation of $\tilde{d}$ and $\tilde{\delta}$. Thus we have

$$\tau^i = \mathfrak{Cl}(d(C^{i-1}))/d(C^{i-1}), \qquad \varrho^i = \mathfrak{Cl}(\delta(C^{i+1}))/\delta(C^{i+1}).$$

Denote also

$$X^i = \mathfrak{Cl}(d(C^{i-1}))/d\delta(C^i), \qquad Y^i = \mathfrak{Cl}(\delta(C^{i+1}))/\delta d(C^i).$$

4.2. Proposition.(a) *There is an exact sequence of $\mathcal{O}$-modules*

$$0 \to \varrho^{i-1} \xrightarrow{\alpha} X^i \xrightarrow{\beta} \tau^i \to 0$$

where the homomorphism α *sends the coset of a class* $x \in \mathfrak{Cl}(\delta(C^{i+1}))$ *into the coset of* dx*; the map* β *is the natural factor-map;* (b) *Similarly, there is an exact sequence*

$$0 \to \tau^{i+1} \xrightarrow{\alpha'} Y^i \xrightarrow{\beta'} \varrho^i \to 0$$

with the map α' *being induced by* δ *and with* β' *being the obvious factor-map;* (c) *For every* i *there is a canonical nondegenerate pairing*

$$\{\ ,\ \} : \tau^i \times \varrho^{i-1} \to \mathcal{M}/\mathcal{O},$$

where $\mathcal{M}$ *denotes the field of fractions of* $\mathcal{O}$*, i.e., the field of germs of meromorphic curves; this pairing is* $\mathcal{O}$*-linear with respect to the first variable and* $\mathcal{O}$*-antilinear with respect to the second variable. In particalar,* τ^i *and* ϱ^{i-1} *are isomorphic as* $\mathcal{O}$*-modules.*

Proof. To prove (a), note that the kernel of β is $\delta(C^{i-1})/\delta d(C^i)$ and so it is enough to show that the homomorphism d induces an isomorphism

$$\varrho^{i-1} = \mathfrak{Cl}(\delta C^i)/\delta C^i \to d(C^{i-1})/d\delta(C^i).$$

From Theorem 3.2 it follows that it is an epimorphism. To show that it is a monomorphism, suppose that $x \in C^{i-1}$ satisfies $t^k x = \delta x'$ and $dx = d\delta y$ for some x', $y \in C^i$ and $k \geq 1$. Then $d\delta(x' - t^k y) = 0$ which implies that $\delta(x' - t^k y) = 0$ and thus $x = \delta y$ represents zero element in ϱ^{i-1}.

Statement (b) follows similarly.

To construct the form

$$\{\ ,\ \} : \tau^i \times \varrho^{i-1} \to \mathcal{M}/\mathcal{O},$$

suppose that we are given $x \in \mathfrak{Cl}(d(C^{i-1})) \subset C^i$ and $y \in \mathfrak{Cl}(\delta(C^i)) \subset C^{i-1}$. Then $t^k x = dx'$ for some $x' \in C^{i-1}$. We define

$$\{x, y\} = t^{-k}(x', y) \in \mathcal{M}/\mathcal{O}, \tag{12}$$

where we use the scalar product (3). Obviously, $\{x, y\}$ depends only on the class of y in ϱ^{i-1}. If $t^l y = \delta y'$, where $y' \in C^i$, then $\{x, y\} = t^{-l}(x, y')$. From this formula it follows that $\{x, y\}$ depends only on the coset of x in τ^i. Thus the form (12) is correctly defined.

Let us show that it is nondegenerate. Suppose that $x \in \mathfrak{Cl}(d(C^{i-1}))$ and $\{x, y\} = 0 \in \mathcal{M}/\mathcal{O}$ for any $y \in \varrho^{i-1}$. Write $t^k = dx'$, where $x' \in C^{i-1}$. We may assume that the number k is as small as possible. Using the decomposition of Theorem 3.2 we may arrange that $x' \in \mathfrak{Cl}(\delta(C^i))$. From our assumptions it follows that $t^{-k}(x', y)$ is a holomorphic curve for any $y \in \mathfrak{Cl}(\delta(C^i))$. In particular, we obtain that the curve $t^{-k}(x', x')$ is holomorphic. If (x', x') is divisible by t in $\mathcal{O}$, then x' is divisible by t in C^{i-1}. Thus, from the minimality of k it follows that $k = 0$ and so x represents zero in τ^i.

Similar arguments show that if $y \in \varrho^{i-1}$ satisfies $\{x, y\} = 0$ for any $x \in \tau^i$, then $y = 0$. This completes the proof.

We need to determine the type of the extensions (over $\mathcal{O}$) which appear in Proposition 4.2. We will show that they are "maximally" nontrivial. In a more precise manner it is expressed in the following statement; here $\mu(X)$ denotes the number of cyclic modules in the decomposition of the $\mathcal{O}$-module X.

4.3. Proposition. *The following equalities hold:*

$$\mu(X^i) = \mu(\tau^i) = \mu(\varrho^{i-1}), \qquad \mu(Y^i) = \mu(\tau^{i+1}) = \mu(\varrho^i).$$

Proof. Let us start from a general remark. As it is well-known, given an extension of $\mathcal{O}$-modules

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0,$$

its type is completely determined by the linear map

$$f : {}^tC \to A_t,$$

where ${}^tC$ denotes the set of all $c \in C$ with $tc = 0$, while A_t denotes A/tA. The map f is defined as $\alpha^{-1}(t\beta^{-1}(c)$ for $c \in C$; it is correctly defined.

Consider the extension, which appear in Proposition 4.2(a). The corresponding linear map

$$f : {}^t(\tau^i) \; \to \; (\varrho^{i-1})_t$$

act as follows. Suppose that $x \in \mathfrak{Cl}(d(C^{i-1}))$ is an element with $tx = dx'$, where $x' \in C^{i-1}$. In fact by Theorem 3.2 we may assume that

$x' \in \mathfrak{Cl}(\delta(C^i))$. We will also assume that x represents a nonzero element in τ^i and thus the scalar product (x', x') is not divisible by t in C^{i-1}.

The image of the class $[x] \in {}^t(\tau^i)$ represented by x under the map f is obviously equal to the class in $(\varrho^{i-1})_t$ represented by x'. Thus the product $\{[x], f([x])\}$ is equal to $t^{-1}(x', x') \in \mathcal{M}/\mathcal{O}$; it is nonzero if $[x] \neq 0$ and so f is a monomorphism. On the other hand, by Proposition 4.2.(c) the modules τ^i and ϱ^{i-1} are isomorphic and therefore the vector spaces ${}^t(\tau^i)$ and $(\varrho^{i-1})_t$ have equal dimensions. This shows that the map f is an isomorphism.

Hence the proposition is proved.

4.4. Proof of Theorem 2.8. Suppose that $(C^\infty(\mathcal{E}), d_t)$ is a deformation of an elliptic complex defined for $-\epsilon < t < \epsilon$. Consider the Laplacians $\Delta_t^i : C^\infty(\mathcal{E}_i) \to C^\infty(\mathcal{E}_i)$ where $i = 0, 1, \ldots, N$ and their parametrized spectral decompositions: cf. proof of Theorem 3.2. Suppose that the eigenvalues $\lambda_n(t)$ are numerated in such a way that they satisfy conditions (i), (ii), (iii) introduced in the proof of Theorem 3.2. Then from Hodge theory it follows that for any value of the parameter t the dimension of the cohomology group $H^i(C^\infty(\mathcal{E}), d_t)$ is equal to the number of zero eigenvalues; if t is close to 0, $t \neq 0$, then this number is precisely the rank of the module of harmonic forms Har^i; the eigenvalues $\lambda_n(t)$ with $n > N_0$ are not zero for all $0 < |t| < \delta$.

For $t = 0$ the dimension of the cohomology space $H^i(C^\infty(\mathcal{E}), d_t)$ is equal to the number N; cf. proof of Theorem 3.2. It is equal to $\mu(\mathfrak{Cl}(\mathrm{im}(\tilde{\Delta}))) = \mu(X^i) + \mu(Y^i)$. Using Propositions 4.2 and 4.3 we obtain

$$\mu(X^i) = \mu(\tau^i), \qquad \mu(Y^i) = \mu(\tau^{i+1})$$

and thus

$$\dim H^i(C^\infty(\mathcal{E}), d_0) = \mathrm{rk}_{\mathcal{O}}\, H^i(\mathcal{O}C^\infty(\mathcal{E}), \tilde{d}) + \mu(\tau^i) \ + \mu(\tau^{i+1}).$$

This completes the proof.

5. Behaviour of the analytic torsion

In this section we will study the analytic torsion for general elliptic complexes; in this generality the analytic torsion was first studied by A.S.Schwarz [26].

We here consider analytic torsion of a one-parameter analytic family of elliptic complexes and show that the analytic torsion as function of the parameter has singularities whose type can be described by the Euler number of the deformation defined in 2.7.

5.1. Let $\mathcal{E} = \oplus_{i=0}^{N}\mathcal{E}_i$ be a graded vector bundle over a closed manifold M and let $(C^\infty(M), d)$ be an elliptic complex; cf. 2.1. Here $d = (d^i)$, where $d^i : C^\infty(\mathcal{E}_i) \to C^\infty(\mathcal{E}_{i+1})$ is a first order differential operator. The *analytic torsion* of this elliptic complex is a positive real number $\rho \in \mathbb{R}_+$ defined as follows.

Fix a Riemannian metric on M and a Hermitian scalar product on each vector bundle $\mathcal{E}_i$; this defines the usual L_2-scalar product in the spaces $C^\infty(\mathcal{E}_i)$ of smooth sections. Let $\delta^i : C^\infty(\mathcal{E}_{i+1}) \to C^\infty(\mathcal{E}_i)$ denote the dual of d^i. The Laplacians

$$\Delta_i = \delta^i d^i + d^{i-1}\delta^{i-1} : C^\infty(\mathcal{E}_i) \to C^\infty(\mathcal{E}_i), \qquad i = 0, 1, \dots, N$$

are elliptic self-adjoint non-negative operators. If $\lambda_{i,1}, \lambda_{i,2}, \dots, \lambda_{i,j}, \dots$ are all the *nonzero* eigenvalues of Δ_i, then one considers the zeta-function defined for large $\Re(s)$ by the formula

$$\zeta_i(s) = \sum_{j=1}^{\infty} \lambda_{i,j}^{-s}.$$

Using the asymptotic expansion of the heat kearnel one shows that zeta-function has an anylitic continuation onto the entire complex plane which is a meromorphic function regular at the origin; cf. [10]. Then the analytic torsion ρ is defined as the unique positive root of

$$\ln(\rho) = \frac{1}{2}\sum_{i=0}^{N} (-1)^i \cdot i\zeta_i'(0).$$

Proposition 5.2. (A.S.Schwarz [26]) *The analytic torsion ρ of an acyclic elliptic complex $(C^\infty(\mathcal{E}), d)$ over an odd-dimensional manifold M does not depend on the choice of the Riemannian metric on M and Hermitian metrics on the vector bundles $\mathcal{E}_i$.*

Independence on the Riemannian metric on M in the case of the De Rham complex was proven by Ray and Singer [23 (Theorem 2.1)]. If there is a nontrivial homology, the analytic torsion depends on the metric. The general formulation of this result states that the Ray-Singer metric on the determimant line of the cohomology does not depend on the Riemannian metric on the manifold; compare [4 (III, Th.1.18)] and also [23 (Th. 7.3)].

The corresponding result is also true for general elliptic complexes; note that the new element here is that we vary Hermitian metrics on the vector bundles $\mathcal{E}_i$ as well as the metric on M. However we decided to impose for the sake of simplicity the acyclicity assumption in the above Proposition.

The proof of Proposition 5.2 given below for completeness, repeats essentially the arguments of [23] and [26].

Proof. Suppose that the metrics on M and $\mathcal{E}_i$'s vary within a real parameter u. Then the scalar product in the space of smooth sections $C^\infty(\mathcal{E}_i)$ will depend on u. For sections $s, s' \in C^\infty(\mathcal{E}_i)$ we will denote their new scalar product by $(s, s')_u$ while $(s, s')_0$ will denote the original undeformed scalar product. Then these two are related by

$$(s, s')_u = (A_u s, s')_0,$$

where $A_u : \mathcal{E}_i \to \mathcal{E}_i$ is a zero order self-adjoint positive operator uniquely determined by the variation of the metrics.

The dual differentials will now depend on the parameter u

$$\delta_u^i : C^\infty(\mathcal{E}_{i+1}) \to C^\infty(\mathcal{E}_i), \qquad \text{where} \qquad \delta_u = A_u^{-1} \delta A_u,$$

and so the Laplacians $\Delta_i(u)$ will also depend on u. For large $\Re(s)$ the zeta-function can be written in the form

$$\zeta_i(u, s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}(e^{-t\Delta_i(u)})\, dt$$

Here $e^{-t\Delta_i(u)}$ denotes the heat kernel of the Laplacian $\Delta_i(u)$; for any $t > 0$ it is an infinitely smoothing operator smoothly depending on u.

Consider the function

$$g(u, t) = \sum_{i=0}^N (-1)^i i \operatorname{Tr}(e^{-t\Delta_i(u)}).$$

Then

$$\frac{\partial g(u,t)}{\partial u} = -t \sum_{i=0}^N (-1)^i i \operatorname{Tr}(e^{-t\Delta_i(u)} \dot{\Delta}_i), \qquad \text{where} \qquad \dot{\Delta}_i = \frac{\partial \Delta_i(u)}{\partial u}.$$

Since $\dfrac{\partial \delta_u}{\partial u} = \delta_u X - X\delta_u$ where $X = A_u^{-1}\dot{A}_u$, we have

$$\frac{\partial \Delta(u)}{\partial u} = d\delta_u X - dX\delta_u + \delta_u X d - X\delta_u d,$$

and similarly to [23 (p.152)] one gets

$$\begin{aligned}
\operatorname{Tr}(e^{-t\Delta_i(u)}\delta_u X d) &= \operatorname{Tr}(e^{-t\Delta_{i+1}(u)} d\delta_u X),\\
\operatorname{Tr}(e^{-t\Delta_i(u)} X\delta_u d) &= \operatorname{Tr}(e^{-t\Delta_i(u)}\delta_u dX),\\
\operatorname{Tr}(e^{-t\Delta_i(u)} dX\delta_u) &= \operatorname{Tr}(e^{-t\Delta_{i-1}(u)}\delta_u dX),
\end{aligned}$$

and therefore

$$\frac{\partial g(u,t)}{\partial u} = -t\sum_{i=0}^{N}\operatorname{Tr}(e^{-t\Delta_i(u)}\Delta_i X) = t\frac{d}{dt}(\sum_{i=0}^{N}(-1)^i\operatorname{Tr}(e^{-t\Delta_i(u)}X)).$$

Thus we obtain for large $\Re(s)$ (cf. [2 (§9.6)]) that

$$\frac{\partial \sum_{i=0}^{N}(-1)^i i\zeta_i(u,s)}{\partial u} = M[tF'(t))] = -sM[F(t)],$$

where

$$F(t) = \sum_{i=0}^{N}(-1)^i\operatorname{Tr}(e^{-t\Delta_i(u)}X).$$

Here M denotes the Mellin transform.

According to [10 (Lemma 1.7.7)], the function $F(t)$ has for $t \to 0$ an asymptotic expansion of the form

$$F(t) = \sum_{j=0}^{\infty} e_j t^{j-n/2}.$$

Using Lemma 9.34 of [2] we conclude that $M[F(t)]$ is holomorphic at $s = 0$ and its value there is equal to $e_{n/2}$. Thus we obtain the following infinitesimal variation formula

$$\frac{\partial \ln(\rho(u)^2)}{\partial u} = -e_{n/2}.$$

But if $\dim M$ is odd, then $e_{n/2} = 0$; cf. [10 (Lemma 1.7.7)].

5.3. Theorem. *Suppose that $(C^\infty(\mathcal{E}), d_t)$ is a deformation of an elliptic complex over a closed Riemannian manifold M defined for values of a parameter t in an interval $-\epsilon < t < \epsilon$. Fix a Hermitian metric on the vector bundles $\mathcal{E} = \oplus \mathcal{E}_i$. Let $\rho(t)$ denotes the value of the analytic torsion for each value of the parameter $t \in (-\epsilon, \epsilon)$. Let χ be the Euler number of the deformation defined with respect to the origin $t = 0$ (cf. Definition 2.7 and the remark afterwards). Then the function*

$$\rho(t) \cdot |t|^\chi$$

is real analytic in a neibourhood of the origin $t \in (-\delta, \delta)$. Thus the Euler number χ determines completely the local singularity of the analytic torsion.

Proof. Consider the parametrized spectral decomposition given by theorem 3.9, p. 392 of Kato [12] applied to the Laplacians $\Delta_i(t)$ for $i = 0, 1, \dots, N$. The eigenvalues $\lambda_{i,j}(t)$ (where i denotes the dimension, $i = 0, 1, \dots, N$ and j numerates the eigenvalues $j = 1, 2, \dots$) can be partitioned into three groups: (i) all eigenvalues satisfying $\lambda_{i,j}(0) \neq 0$; (ii) those which vanish for $t = 0$ but are not identically zero; (iii) identically zero eigenvalues. Eigenvalues of the type (iii) do not appear in the zeta-functions and so they do not influence the torsion $\rho(t)$. Thus the i-dimensional zeta-function $\zeta_i(t, s)$ can be represented as a sum of two terms

$$\zeta_i(t, s) = \zeta_i^{(1)}(t, s) + \zeta_i^{(2)}(t, s)$$

incorporating the eigenvalues of type (i) and (ii) correspondingly. The contribution of the first zeta-function $\frac{\partial \zeta_i^{(1)}(t, s)}{\partial s}|_{s=0}$ into the analytic torsion gives a nonzero factor *analytically depending on t*. Here we use the following well-known general fact: If

$$P_t : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E}), \quad t \in (a, b)$$

is a family of invertible elliptic self-adjoint positive pseudo-differential operators of order $m > 0$ (which does not depend on t), then the zeta-function determinant

$$\mathrm{Det}(P_t) = \exp(-\zeta'(0, P_t))$$

is a real analytic function of $t \in (a, b)$. We refer to [9 (p. 373)] and [18] for a proof (even in the situation with weaker assumptions).

For the second zeta-function (which is finite) we have

$$\frac{d}{ds}\zeta_i^{(2)}(t,s)|_{s=0} = -\ln\prod_{j=1}^{N_i}\lambda_{i,j}(t),$$

where $\lambda_{i,1}(t), \lambda_{i,2}(t), ..., \lambda_{i,N_i}(t)$ are all eigenvalues of type (ii). Then we may write

$$\prod_{j=1}^{N_i}\lambda_{i,j}(t) \;=\; c_i(t)\cdot t^{\alpha_i},$$

where $c_i(t)$ is a nonvanishing analitic function, and α_i coincides with the dimension of the following space

$$\dim(\mathfrak{Cl}(\tilde{\Delta}(\mathcal{O}C^\infty(\mathcal{E}_i)))/\tilde{\Delta}(\mathcal{O}C^\infty(\mathcal{E}_i))),$$

(as it was shown in the proof of Theorem 3.2). Now Proposition 4.2 gives

$$\begin{aligned}\alpha_i &= \dim X^i + \dim Y^i\\ &= (\dim\tau^i + \dim\varrho^{i-1}) + (\dim\tau^{i+1} + \dim\varrho^i)\\ &= 2(\dim\tau^i + \dim\tau^{i+1}).\end{aligned}$$

Substituting this into the formula defining the analytic torsion we get

$$\rho(t) \;=\; |t|^{-\chi}F(t),$$

where $F(t)$ is a nonzero function which is analytic in a neigbourhood of $t = 0$.

6. Torsion of spectral sequence of deformation

6.1. Theorem. *Suppose that $(C^\infty(\mathcal{E}), d_t)$ is a deformation of an elliptic complex defined for $(-\epsilon < t < \epsilon)$. Then there exists a spectral sequence E_r^*, $r \geq 1$ with the following properties:*

(1) *The initial term E_1^* of the spectral sequence equals to the cohomology of the original undeformed elliptic complex*

$$E_1^* \;=\; H^*(C^\infty(\mathcal{E}), d_0);$$

(2) *The differentials of the spectral sequence*

$$d_r : E_r^* \to E_r^{*+1}$$

have degree one and depend only on the derivatives of order $1 \leq i \leq r$ *with respect to the parameter* t *of the differential operator*

$$d_t : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E}).$$

In particular, the first differential of the spectral sequence is given by the action of the first derivative $\frac{d(d_t)}{dt}|_{t=0}$ *on the harmonic forms of the undeformed operator* d_0. (3) *For large* r *all the differentials of the spectral sequence vanish and the limit term* E_∞^* *is isomorphic to the cohomology of the elliptic complex* $H^*(C^\infty(\mathcal{E}), d_t)$ *for generic* t *close to the origin* $-\delta < t < \delta$, $t \neq 0$.

Proof. Consider the germ-complex of the deformation; cf. 2.5. For $r \geq 0$ denote by Z_r^i the set of all holomorphic germs $f \in \mathcal{O}C^\infty(\mathcal{E}_i)$ satisfying $\tilde{d}f \in t^r \mathcal{O}C^\infty(\mathcal{E}_{i+1})$. Then, following the usual technique of constructing sepectral sequences, one defines

$$E_r^i \;=\; Z_r^i/(tZ_{r-1}^i + t^{1-r}\tilde{d}Z_{r-1}^{i-1})$$

and the differential

$$d_r : \; E_r^i \;\to\; E_r^{i+1}$$

to be the homomorphism induced by the action of $t^{-r}\tilde{d}$ on Z_r^i. Thus it follows that

$$H^*(E_r, d_r) \simeq E_{r+1}^*,$$

i.e., one gets a spectral sequence.

For $r = 1$ the space Z_1^i consists of germs of curves $f \in \mathcal{O}C^\infty(\mathcal{E}_i)$ such that $f(0) \in \ker[d_0 : \; C^\infty(\mathcal{E}_i) \to C^\infty(\mathcal{E}_{i+1})]$ while Z_0^i coincides with the whole $\mathcal{O}C^\infty(\mathcal{E}_i)$; now, from the definition above we obtain that the initial term E_1^* can be identified with the cohomology of the original undeformed complex.

Suppose that

$$d_t = d_0 + t\sigma_1 + t^2\sigma_2 + \dots$$

is the Taylor expansion of the differential operator (boundary homomorphism) of the elliptic complex, where σ_i belong to $\oplus_j \operatorname{Diff}_1(\mathcal{E}_j, \mathcal{E}_{j+1})$;

i.e., they are first order partial differential operators. Then the action of the differential d_r of the spectral sequence on an element of the group E_r^i, represented as the coset containing some $f \in Z_r^i$, can be found as follows. Let $f(t) = f_0 + tf_1 + t^2 f_2 + \dots$ be the Taylor expansion of $f(t)$. Then (since $f \in Z_r^i$)) the following r equations

$$d_0 f_i + \sum_{j=1}^{i} \sigma_j f_{i-j} = 0, \qquad \text{for} \qquad i = 0, 1, \dots, r-1$$

have to hold. Note that the coset of f in E_r^i depends only on the first term f_0 of the above expansion (since any element f in Z_r^i with $f_0 = 0$ would belong to tZ_{r-1}^i). According to the definition above, the image of the coset represented by f under the differential d_r is represented by an element of Z_r^{i+1} having the first term

$$\sum_{j=1}^{r} \sigma_j f_{r-j}.$$

Thus we see that only the operators σ_j with $j = 1, 2, \dots, r$ take part in the description of d_r. In the case $r = 1$ the above formula becomes just $\sigma_1 f_0$.

The finiteness statement of the theorem follows from the fact that the initial term is finite dinemsional.

To understand the structure of the limit of our spectral sequence we are going to apply again the theorem on perturbations of self-adjoint operators due to T. Kato [12 (p. 392)]. As it has been mentioned in the proof of Theorem 3.2, the parametrized spectral decomposition $\{\phi_n(t), \lambda_n(t)\}_{n \geq 1}$ given by this theorem may contain only finitely many terms with $\lambda_n(t) \equiv 0$. We may assume that the eigenfunctions and eigenvalues are numerated in such a way that $\lambda_n(t) \equiv 0$ for $1 \leq n \leq N$ and $\lambda_n(t) \not\equiv 0$ for $n > N$. Then for r large we easily see that any $f \in Z_r^i$ must belong to the free $\mathcal{O}$-submodule A^i of $\mathcal{O}C^\infty(\mathcal{E}_i)$ generated by $\phi_n(t)$'s with $1 \leq n \leq N$. Then from the definition of E_r^i we get that for large r the group E_r^i is precisely A^i/tA^i; thus the limit term E_∞^i can be identified with the space of harmonic forms of dimension i for all generic values of the parameter t close to 0, $t \neq 0$.

6.2. Remark. The above spectral sequence can be also described entirely in terms of the cohomology of the germ-complex; cf.

2.5. If F^i denotes the free part of the i-dimensional cohomology module $H^i(\mathcal{O}C^\infty(\mathcal{E}), \tilde{d})$ and τ^i denotes its torsion part, then the r-th term E_r^i of the spectral sequence is isomorphic to

$$E_r^i \simeq F^i/tF^i \ \oplus \ \tau^i/[t\tau^i + (\tau^i)_{r-1}] \ \oplus \ (\tau^{i+1})_r/(\tau^{i+1})_{r-1}.$$

Here we use the following notation: for a module τ over the ring $\mathcal{O}$ and for a positive integer r the symbol $(\tau)_r$ denotes $\{a \in \tau; t^r a = 0\}$.

The differential $d_r : E_r^i \to E_r^{i+1}$ vanishes on the first and the second summands of the above decomposition and maps the third summand of the decomposition of E_r^i into the second summand of the decomposition of E_r^{i+1} via the obvious homomorphism

$$(\tau^{i+1})_r/(\tau^{i+1})_{r-1} \ \to \ \tau^{i+1}/[t\tau^{i+1} + (\tau^{i+1})_{r-1}]$$

induced by the inclusions.

This remark together with some elementary calculations gives the following.

6.3. Proposition. *In the situation of Theorem 6.1 suppose that χ'_r denotes the secondary Euler characteristics of the term E_r^* of the above spectral sequence, i.e.,*

$$\chi'_r \ = \ \sum_{i=1}^{\infty}(-1)^i i \dim E_r^i, \qquad \textit{where} \quad r = 1, 2, 3, \ldots, \infty.$$

Then the Euler number χ of the deformation of the elliptic complex equals

$$\chi \ = \ \sum_{r=1}^{\infty}[\chi'_r - \chi'_\infty].$$

The last formula we understand as $\sum_{r=1}^{N}[\chi'_r - \ \chi'_N]$, where N is a sufficiently large number.

6.4. In the next subsection we are going to define a positive real number θ which measures *the torsion* of the above spectral sequence.

First, let us recall a few general facts about determinants and volume forms and fix notation. If V is a vector space, $\Lambda(V)$ will denote the *determinant line* of V which is $\Lambda^n(V)$, where $n = \dim V$. Nonzero elements of the determinant line $\Lambda(V)$ are called *volume forms* on V. If $a, b \in \Lambda(V)$ are two volume forms, then $a = \lambda b$ for some nonzero number λ; this relation we will write as $\lambda = a/b$.

The determinant line of a graded vector space $V = \oplus V_j$ (we assume here that only finitely many of V_j's are nonzero) is defined as

$$\Lambda(V) = \otimes \Lambda(V_j)^{(-1)^j};$$

the inverse of a 1-dimensional vector space is understood as the dual space, and the product in the previous formula denotes the tensor product. The determinant line of a direct sum of graded vector spaces is canonically isomorphic to the product of the determinant lines; determinant line of a zero vector space is defined to be $\mathbb{C}$.

If V is a cochain complex and H is its cohomology, there is a *canonical* isomorphism

$$\phi_V : \Lambda(V_*) \to \Lambda(H_*).$$

We will describe the homomorphism ϕ_V in the case where the complex V has the form

$$0 \to V_{j-1} \xrightarrow{d} V_j \to 0$$

with d an isomorphism. Let $v_i \in \Lambda(V_i)$ be two volume forms, where $i = j-1,\ j$. Let $v_i^* \in \Lambda(V_i^*) = \Lambda(V_i)^*$ be the dual volume form. Then

$$\phi_V(v) \;=\; \det(d)^{(-1)^j} \cdot 1,$$

where $1 \in \mathbb{C}$ is the canonical element in $\mathbb{C} = \Lambda(H)$, and $\det(d)$ is the determinant of any matrix representing the homomorphism d with respect to a pair of basises of V_{j-1} and V_j realizing the volume forms v_{j-1} and v_j respectively. Here v denotes the following element of the determinant line $\Lambda(V)$:

$$v \;=\; \begin{cases} v_{j-1}v_j^*, & \text{if } j \text{ is odd,} \\ v_{j-1}^* v_j, & \text{if } j \text{ is even.} \end{cases}$$

6.5. Let $\Lambda(E_r^*)$ denote the determinant line of the spectral sequence E_r constructed in 6.1 considered as a graded vector space:

$$\Lambda(E_r^*) \;=\; \otimes_{j\geq 0}\Lambda(E_r^j)^{(-1)^j}.$$

As it was explained in 6.4, for any r there is a canonical isomorphism

$$\phi_r : \Lambda(E_r^*) \to \Lambda(E_{r+1}^*).$$

Since the initial term E_1 of our spectral sequence coincides with the space of harmonic forms of the undeformed elliptic complex $(C^\infty(\mathcal{E}), d_0)$, there is the canoninal volume form $c_1 \in \Lambda(E_1^*)$ (defined up to multiplication by a scalar with absolute value 1) which is represented as the wedge-product of harmonic forms. The limit term of the spectral sequence also has a canonical volume element $c_\infty \in \Lambda(E_\infty^*)$; it can be represented as the wedge product of harmonic forms of $(C^\infty(\mathcal{E}), d_t)$ for generic nonzero t close to 0 if one identifies the limit term E_∞^* accoding to Theorem 6.1. Thus for large r the positive real number

$$\theta = |[\phi_r \circ \cdots \circ \phi_1(c_1) : c_\infty]| \in \mathbb{R}_+$$

does not depend on r. We will call θ *torsion* of the above spectral sequence.

6.6. Theorem. *Under the assumptions of theorem 5.3 the limit* $\lim_{t\to 0} \rho(t)|t|^\chi$ *is equal to* $\theta \cdot \rho(0)$ *where* θ *denotes the torsion of the spectral sequence associated with the deformation, and* $\rho(0)$ *is the value of the analytic torsion of the elliptic complex* $(C^\infty(\mathcal{E}), d_t)$ *for* $t = 0$.

Proof. Let us use the notation introduced in the proof of theorem 5.3. The arguments described there show that the limit $\lim_{t\to 0} \rho(t)|t|^\chi$ is equal to

$$\rho(0) \cdot \lim_{t\to 0} [|t|^\chi \cdot \prod \lambda_{i,j}(t)^{(-1)^{i+1}i/2}],$$

where the product it taken over all nonzero eigenvalues of type (ii) (cf. proof of Theorem 5.3) of the Laplacians. If we write

$$\lambda_{i,j}(t) = t^{\nu_{i,j}} \overline{\lambda}_{i,j}(t), \qquad \text{where} \quad \overline{\lambda}_{i,j}(0) \neq 0, \quad \nu_{i,j} > 0,$$

then to prove the Theorem we have to show that the torsion of the spectral sequence θ is given by

$$\theta = \prod \overline{\lambda}_{i,j}(0)^{(-1)^i i/2}.$$

To do this we are going to use the duality relations of Proposition 4.2. But first let us introduce the following notation.

Let $h_j \in \mathcal{O}C^\infty(\mathcal{E}_{i_j})$ denote a basis over $\mathcal{O}$ for the space of harmonic forms of the germ-complex, $\tilde{\Delta} h_j = 0$. It is a finite set; we may construct h_j's in such a way that they are orthonormal with respect to the scalar product (3).

Let $f_j \in \mathcal{O}C^\infty(\mathcal{E}_{i_j})$ denote a finite set of germs satisfying the following conditions:

(a) they are orthonormal with respect to the scalar product (3);

(b) they are eigenvectors of the operator $\tilde{d}\tilde{\delta}$:

$$\tilde{d}\tilde{\delta}(f_j) = \mu_j f_j \qquad \text{where} \qquad \mu_j \in \mathcal{O}, \quad \mu_j \neq 0, \quad \mu_j(0) = 0;$$

in particular, all f_j belong to $\mathfrak{Cl}(\tilde{d}\tilde{\delta}(\mathcal{O}C^\infty(\mathcal{E}_{i_j})))$;

(c) for any fixed i the cosets of $\{f_j | i_j = i\}$ generate the $\mathcal{O}$-module

$$X^i = \mathfrak{Cl}(\tilde{d}\tilde{\delta}(\mathcal{O}C^\infty(\mathcal{E}_i)))/\tilde{d}\tilde{\delta}(\mathcal{O}C^\infty(\mathcal{E}_i)).$$

The existence of such set of curves f_j follows easily from the parametrized spectral decomposition; cf. §3. Now, we can write

$$\mu_j(t) \;=\; t^{\nu_j}\overline{\mu}_j(t), \qquad \text{where} \quad \nu_j > 0 \quad \text{and} \quad \overline{\mu}_j(0) \neq 0.$$

Note that all numbers ν_j are *even* and $\overline{\mu}_j(0)$ are *positive*; this fact follows from the positivity of the Laplacians.

Let g_j denote

$$g_j = (\mu_j)^{-1/2}\tilde{\delta}f_j.$$

Here the existence of a smooth square-root $\mu_j^{1/2}$ follows from the positivity mentioned above. In fact g_j is a smooth curve, i.e., an element of $\mathcal{O}C^\infty(\mathcal{E}_{i_j-1})$; it follows from $(g_j, g_j) = 1$ where the scalar product here is given by (3). On the other hand $\tilde{\delta}\tilde{d}g_j = \mu_j g_j$. Thus the set of g_j's forms an orthomormal family, which is orthogonal to $\{h_j\}$ and $\{f_j\}$. Moreover the cosets of g_j's with the dimension $i_j = i$ fixed generate the $\mathcal{O}$-module

$$Y^{i-1} = \mathfrak{Cl}(\tilde{\delta}\tilde{d}(\mathcal{O}C^\infty(\mathcal{E}_{i-1})))/\tilde{\delta}\tilde{d}(\mathcal{O}C^\infty(\mathcal{E}_{i-1})).$$

Thus we obtain that the germs $\{f_j, g_j\}$ comprise all eigenvalues of type (ii) and that each eigenvalue μ_j appears twice - once in dimension i_j as the eigevalue of f_j and another time in dimension $i_j - 1$ as the eigenvalue of h_j. Hence, from the remark in the beginning of the proof it follows that to finish the proof we are left to show that

$$\theta \;=\; \prod \overline{\mu}_j(0)^{(-1)^{i_j}}. \tag{13}$$

Consider now the spectral sequence E_r. Its initial term E_1 coincides with the set of harmonic forms of the undeformed elliptic complex. Thus we may think of E_1 as being the vector space generated by $\{h_j, f_j, g_j\}$ which we will view now as abstract symbols. Since the differential d_r of the spectral sequence is given by $t^{-r}\tilde{d}$ we obtain the following:

(a) according to the construction of the spectral sequence desribed in the proof of Theorem 6.1, the differential d_r vanishes on all elements h_j and f_j for all r;

(b) the action of the differential d_r on an element of the form g_j is given by the formula:

$$d_r(g_r) \;=\begin{cases} 0, & \text{if } 2r < \nu_j, \\ \overline{\mu}_j(0)^{1/2} f_j, & \text{if } 2r = \nu_j. \end{cases}$$

In order to justify the last formula, note that

$$d_r(g_j) = t^{-r}\tilde{d}g_j|_{t=0} = t^{-r}\tilde{d}\mu_j(t)^{-1/2}\tilde{\delta}f_j|_{t=0} = (t^{-r}\mu_j^{1/2})|_{t=0}f_j.$$

Thus, we see that the term E_r of the spectral sequence is the vector space spanned by all symbols $\{h_j\}$ and by the symbols

$$\{f_j, g_j| \quad \text{with} \quad \nu_j \geq 2r\}.$$

Note also that each term of our spectral sequence has a natural metric, and the basis above is in fact orthonormal with respect to this metric. Thus one easily obtain (13) by computing the torsion of the spectral sequence by counting the determinants.

7. Deformations of flat vector bundles

7.1. Let M be a closed Riemannian manifold of dimension n and let $\mathcal{E}$ be a Hermitian vector bundle over M. It is well-known that to specify a flat structure on $\mathcal{E}$ (i.e., a representation of $\mathcal{E}$ as a vector bundle with descrete structure group) it is equivalent to fix a flat linear connection on $\mathcal{E}$

$$\nabla : A^k(M;\mathcal{E}) \to A^{k+1}(M;\mathcal{E}), \quad k = 0, 1, 2, ..., \qquad \nabla^2 = 0.$$

We want to emphisize that considering flat Hermitian vector bundles, we do not suppose in general that the connection ∇ preserves the

Hermitian structure on $\mathcal{E}$. ∇ is called *unitary* connection if it preserves the Hermitian metric on $\mathcal{E}$, or, in other words, if the Hermitian metric on $\mathcal{E}$ if *flat.*

7.2. By a *deformation of the flat vector bundle* consisting of the data $(\mathcal{E}, \nabla)$ described above, we will understand a one-parametric family ∇_t of flat linear connections,

$$\nabla_t : A^k(M;\mathcal{E}) \to A^{k+1}(M;\mathcal{E}), \quad k = 0, 1, 2, ..., \qquad \nabla_t^2 = 0$$

defined in a neighbourhood of zero $-\epsilon < t < \epsilon$ and satisfying $\nabla_0 = \nabla$. The Hermitian structure on $\mathcal{E}$ is supposed to be independent of t. The deformation ∇_t is said to be *unitary* if all connections ∇_t are unitary.

The precise value of the number ϵ will be of no importance to us; thus we are actually interested in studying *germs* of the one-parametric families ∇_t.

Any such deformation determines in an obvious way a deformation of the twisted De Rham complex

$$\cdots \to A^k(M;\mathcal{E}) \xrightarrow{\nabla_t} A^{k+1}(M;\mathcal{E}) \xrightarrow{\nabla_t} A^{k+2}(M;\mathcal{E}) \to \dots$$

which can be considered as an instance of deformations of elliptic complexes as in §2.2. Our goal in this section is to review a result of [8] which shows that in the case of deformations of twisted De Rham complex the general notions of $\mathcal{O}$-cohomology, the Euler number of the deformation χ and others can be expressed through purely *homological* invariants determined by the deformation of the monodromy representation. We will also observe that there are some duality relations for the torsion part of the $\mathcal{O}$-cohomology.

7.3. Deformation of the monodromy representation. Suppose that a deformation of flat vector bundle $(\mathcal{E}, \nabla_t)$, $t \in (-\epsilon, \epsilon)$ is given. Fix a base point $x \in M$.

For any value of the parameter t the flat connection ∇_t determines the monodrony representation

$$\kappa_t : \pi = \pi_1(M, x) \to \mathrm{GL}_{\mathbb{C}}(\mathcal{E}_x), \qquad -\epsilon < t < \epsilon, \tag{14}$$

where $\mathcal{E}_x$ is the fiber above x. It is defined by the usual precedure of "analytic continuation" along loops which start and finish at x.

Let V denote the set of germs of holomorphic curves in $\mathcal{E}_x$ (observe, that $V = \mathcal{O}\mathcal{E}_x$ in the notation of §2.3). V is a free $\mathcal{O}$-module of rank

$m = \dim \mathcal{E}_x$. Then the family of representations (14) defines a single representation

$$\kappa : \pi \to \mathrm{GL}_{\mathcal{O}}(V). \tag{15}$$

which is given by

$$(\kappa(g) \cdot \alpha)(t) = \kappa_t(g)(\alpha(t)), \qquad -\epsilon < t < \epsilon$$

for $g \in \pi$, $\alpha \in V = \mathcal{O}\mathcal{E}_x$, $\alpha : (-\epsilon, \epsilon) \to \mathcal{E}_x$.

Thus, via the homomorphism (15) one may view V as a left $\mathcal{O}[\pi]$-module; this module V we call *deformation of the monodromy representation* determined by the deformation of the flat bundle.

On the other hand, one may view V as a local coefficients system over M and consider the cohomology of this local system. Recall that the cohomology of M with coefficients in V is by definition

$$H^i(M; V) = H^i(\mathrm{Hom}_{\mathbb{C}[\pi]}(C_*(\tilde{M}), V)),$$

where $C_*(\tilde{M})$ is the singular (or simplicial) chain complex of the universal covering $\tilde{M}$ of M. The fundamental group π acts freely on $\tilde{M}$ from the left. This cohomology $H^i(M; V)$ is a finitely generated mudule over $\mathcal{O}$.

7.4. Theorem ([8]). *There is a canonical $\mathcal{O}$-isomorphism between the cohomology $H^i(M, V)$ of the local system determined by the deformation of the monodromy representation and the cohomology of the germ-complex (cf. §2.5) of the deformation of the twisted De Rham complex.*

This statement is a version of the De Rham theorem, and the standard proof using fine resulutions of sheaves can be adopted to this case. This is shown in the proof of Proposition 4.5 in [8], where it is assumed that the deformation ∇_t is unitary. This assumption (although important for other purposes of the paper [8]) is irrelevant for Proposition 4.5 and is not used in its proof.

7.5. Let $\Lambda = \mathbb{C}[\pi]$ denote the group ring. V can be considered as a $\Lambda - \mathcal{O}$-bimodule (i.e., as left Λ-module and right $\mathcal{O}$-mudule) and by Theorem 7.4 this structure determines the cohomology of the germ-complex. One can describe V as $\Lambda - \mathcal{O}$-bimodule in the following way. V has an infinite decreasing filtration by $\Lambda - \mathcal{O}$-submodules

$$V \supset V_1 \supset V_2 \supset V_3 \supset \dots, \qquad \text{where} \qquad V_j = t^j V$$

such that all factors

$$V_j/V_{j+1} \simeq \mathcal{E}_x$$

are identical and isomorphic to the original (undeformed) monodromy representation, i.e., the monodromy representation corresponding to ∇_0. Thus V can be obtained as the result of a chain of extensions and the way how these extensions are built is determined by the actual type of the deformation.

Suppose that V is flat as a Λ-module (this assumption is actually satisfied in many examples). Then one can compute the $\mathcal{O}$-torsion submodules τ^i of $H^i(M;V)$ in the following way. Denote $\tau^i_{(r)} = \ker[t^r : H^i(M;V) \to H^r(M;V)]$. Then $H^i(M;V) = H^i(M;\Lambda) \otimes_\Lambda V$ and

$$\tau^i_{(r)} \simeq \mathrm{Tor}^1_\Lambda(H^i(M;\Lambda); V/t^rV). \tag{16}$$

For large r this gives a formula for τ^i. For $r = 1$ we obtain

$$\tau^i_{(1)} \simeq \mathrm{Tor}^1_\Lambda(H^i(M;\Lambda); \mathcal{E}_x), \tag{17}$$

where $\mathcal{E}_x$ is regarded as a left Λ-module via the monodromy representation of ∇_0. This shows, in particular, that $\tau^i_{(1)}$ depends only on ∇_0 and not on the deformation.

7.6. Proposition. *For a unitary deformation of a flat vector bundle $(\mathcal{E}, \nabla_t)$, $t \in (-\epsilon, \epsilon)$ over a closed n-dimensional manifold M there are isomorphisms of $\mathcal{O}$-modules*

$$\mathrm{Har}^i \simeq \mathrm{Har}^{n-i}, \qquad \tau^i \simeq \tau^{n-i+1}, \qquad i = 1, 2, \dots. \tag{18}$$

In particular, it follows that if the dimension n is even, then the Euler number χ of a unitary deformation of a flat vector bundle over M vanishes. For n odd, $n = 2l - 1$, the Euler number of a unitary deformation is equal to

$$\chi = (-1)^l \dim \tau^l + 2 \sum_{i=1}^{l-1} (-1)^i \dim \tau^i, \quad \textit{and thus} \quad \chi \equiv \tau^l \ (\bmod 2). \tag{19}$$

Note that the fact that $\chi = 0$ if n is even and the deformation is unitary follows from Theorem 5.3 and the theorem of Ray and Singer [RS] stating that in this situation the analytic torsion vanishes.

Proof. Consider the parametrized Hodge decomposition

$$(20)\quad \mathcal{O}A^i(M;\mathcal{E}) = \mathrm{Har}^i \oplus \mathfrak{Cl}(\tilde{d}(\mathcal{O}A^{i-1}(M;\mathcal{E}))) \oplus \mathfrak{Cl}(\tilde{\delta}\mathcal{O}A^{i+1}(M;\mathcal{E}))$$

given by Theorem 3.2. Because of the formulars

$$(21)\qquad ** = (-1)^{i(n-i)} \qquad \text{and} \qquad \tilde{\delta} = (-1)^{n(i+1)+1} * \tilde{d}*,$$

the star operator $*$ maps Har^i onto Har^{n-i}; on the other hand $*$ applied to the second term $\mathfrak{Cl}(\tilde{d}(\mathcal{O}A^{i-1}(M;\mathcal{E})))$ maps it onto the third term $\mathfrak{Cl}(\tilde{\delta}\mathcal{O}A^{n-i+1}(M;\mathcal{E}))$ isomorphically and vise versa. Thus the star operator induces isomorphisms $\tau^i \simeq \varrho^{n-i}$ which together with (c) of Proposition 4.2 prove our statement.

Note that the Hodge star operator

$$* : \mathcal{O}A^i(M;\mathcal{E}) \to \mathcal{O}A^{n-i}(M;\mathcal{E})$$

is defined on the curves pointwise: $(*\alpha)(t) = *(\alpha(t))$ for $\alpha : (-\epsilon,\epsilon) \to A^i(M;\mathcal{E})$ and $t \in (-\epsilon,\epsilon)$. It depends only on the Riemannian metric on M.

7.7. Dual connection. The material of this subsection is related to [2 (p.121)], and [5 (p. 62)]. Consider a *non-unitary* flat connection ∇ on a Hermitian vector bundle $\mathcal{E}$ over M. Then the first of the relations (21) still holds while the second fails. More precisely, the scalar product on the space of differential forms with values in $\mathcal{E}$ is given by

$$(\omega,\omega') = \int_M \omega \wedge *\omega',$$

where

$$\wedge : A^i(M;\mathcal{E}) \otimes A^j(M;\mathcal{E}) \to A^{i+j}(M)$$

is the Hermitian wedge-product which uses the Hermitian metric $\mathcal{E} \otimes \mathcal{E} \to \mathbb{C}$ as the bundle map. Let $\mathrm{Herm}(\mathcal{E})$ denote the vector bundle of self-adjoint endomorphisms of $\mathcal{E}$; it is a subbundle of $\mathrm{End}(\mathcal{E})$. The covariant derivative of the Hermitian metric on $\mathcal{E}$ is a 1-form on M with values in the bundle of Hermitian metrics on $\mathcal{E}$; it is however more convenient to use the given metric on $\mathcal{E}$ to view this covariant derivative as a 1-form

$$\nu \in A^1(M;\mathrm{Herm}(\mathcal{E})).$$

This form ν is defined by the property that

$$d(s, s') = (\nabla s, s') + (s, \nabla s') + \nu(s) \wedge s' \tag{22}$$

holds for any pair of sections s, s' of $\mathcal{E}$.

Then for forms $\omega, \omega' \in A^*(M; \mathcal{E})$ with values in $\mathcal{E}$ we have

$$d(\omega \wedge \omega') = \nabla\omega \wedge \omega' + (-1)^{|\omega|}\omega \wedge \nabla\omega' + \nu(\omega) \wedge \omega', \tag{23}$$

where $\nu(\omega) \in A^{|\omega|+1}(M; \mathcal{E})$ is obtained by the wedge product of differential forms ν and ω, which uses the canonical pairing of vector bundles $\mathrm{Herm}(\mathcal{E}) \otimes \mathcal{E} \to \mathcal{E}$. Note also that the last term in (23) can be written as

$$\nu(\omega) \wedge \omega' = (-1)^{|\omega|}\omega \wedge \nu(\omega').$$

Thus, $\nabla' = \nabla + \nu$ is a connection on $\mathcal{E}$; we will call ∇' the *connection dual to* ∇.

Suppose that ∇ is flat. Then locally, over an open set $U \subset M$ the vector bundle $\mathcal{E}$ has a basis consisting of flat sections $s_1, s_2, \dots, s_m$; let us write

$$(s_i, s_j) = h_{ij} \qquad \text{and} \qquad \nu(s_i) = \sum_j \omega_{ij} s_j,$$

where $H = (h_{ij})$ and $\Upsilon = (\omega_{ij})$ are matrices of functions and 1-forms on U respectively. Thus (22) gives

$$\Upsilon = H^{-1} \circ dH. \tag{24}$$

The dual connection is also flat $(\nabla')^2 = 0$. To show this we may use local coordinate system as above. The curvature of the dual connection ∇' is represented by the matrix of 2-forms $d\Upsilon + \Upsilon^2$; the fact that the last matrix is zero follows easily from (24).

This procedure of finding the dual connection is involutive: the connection dual to ∇' is the original connection ∇.

Another property of the dual connection: a flat connection ∇ is unitary if and only if it coincides with its dual ∇', i.e., $\nabla = \nabla'$.

Using the dual connection we may rewrite (23) in the form

$$d(\omega \wedge \omega') = \nabla\omega \wedge \omega' + (-1)^{|\omega|}\omega \wedge \nabla'(\omega'). \tag{25}$$

The equality (25) implies that *the adjoint of the connection* ∇^* (which is defined by the identity $(\nabla^*\omega, \omega') = (\omega, \nabla\omega')$) *can be expressed through the star operator and the dual connection by the formula*

$$(26) \qquad \nabla^*\omega = (-1)^{n(p+1)+1} * \nabla' * \omega \quad \text{for} \quad \omega \in A^p(M;\mathcal{E}).$$

There is also a similar formula for the adjoint of the dual connection:

$$(27) \qquad \nabla'^*\omega = (-1)^{n(p+1)+1} * \nabla * \omega \quad \text{for} \quad \omega \in A^p(M;\mathcal{E}).$$

General duality relations. Suppose that $(\mathcal{E}, \nabla_t)$ is a deformation of a flat vector bundle over an n-dimensional manifold M, where $t \in (-\epsilon, \epsilon)$. Consider the family of dual connections ∇'_t. We will denote by τ^i and τ'^i the $\mathcal{O}$-torsion submodules of the original and the dual deformations correspondingly. Similarly, χ and χ' will denote the Euler numbers of the above deformations and so on.

7.8. Proposition. *The following $\mathcal{O}$-isomorphisms hold*

$$\mathrm{Har}^i \simeq \mathrm{Har}'^{n-i}, \qquad \tau^i \simeq \tau'^{n-i+1}$$

for all $i = 1, 2, \ldots$ In particular, $\chi = (-1)^{n+1}\chi'$.

Proof. This follows from relations (26) and (27) and the parametrized Hodge decomposition of Theorem 3.2.

8. Some examples

8.1. Example 1: Alexander modules. Here we will consider deformations of flat line bundles whose monodromy representations have some "integrality" property. More precisely, we will consider monodromy representations which factor through a fixed homomorphism onto an infinite cyclic group. In this situation any analytic curve on the punctured plane $\mathbb{C}^*$ determines naturally an analytic family of representations. We will show that in this case the torsion modules of the $\mathcal{O}$-cohomology modules can be described in terms of the Alexander modules.

8.1.1. Let M be a closed Riemanian manifold of dimension n and let $\phi : \pi_1(M) \to J$ be a fixed epimorphism. Here $J = (a)$ denotes an infinite cyclic group (written multiplicatively) with generator a. Given a point on the punctured complex plane $\beta \in \mathbb{C}^* = \mathrm{GL}(1)$, one constructs a 1-dimensional representation of J such that $a \mapsto \beta$; composed with ϕ it gives a representation of $\pi_1(M)$. Therefore, any analytic curve $\beta : (-\epsilon, \epsilon) \to \mathbb{C}^*$ determines analytic family of representations

$$\kappa_t : \pi_1(M) \to \mathrm{GL}(1) = \mathbb{C}^*, \qquad t \in (-\epsilon, \epsilon),$$

by the rule

$$\kappa_t(g) = \beta(t)^l, \qquad \text{if} \quad \phi(g) = a^l, \qquad \text{where} \quad g \in \pi_1(M).$$

We will denote $\xi = \beta(0)$ and will assume that the derivative $\beta'(0) \neq 0$ is nonzero.

According to our general construction in §7.3, the above family of representations determines a single representation

$$\kappa : \pi_1(M) \to \mathrm{GL}_{\mathcal{O}}(1) = \mathcal{O}^*.$$

The latter can be described as follows. Let $\Lambda = \mathbb{C}[J] = \mathbb{C}[a, a^{-1}]$ be the group ring of J with complex coefficients. Consider the ring homomorphism $\psi : \Lambda \to \mathcal{O}$ which is defined by $a \mapsto \beta(t)$. Then $\kappa = \psi \circ \phi$.

The homomorphism ψ allows to regard $\mathcal{O}$ as a Λ-module; this module we will denote $\mathcal{O}_\beta$ since it depends on the choice of the curve β. In the following lemmas we will observe some homological properties of $\mathcal{O}_\beta$.

8.1.2. Lemma. *$\mathcal{O}_\beta$ is flat as a Λ-module.*

Proof. First note that ψ is a monomorphism. On the other hand the ring $\mathcal{O}$ has no zero divisors and thus $\mathcal{O}_\beta$ has no Λ-torsion. Since Λ is a principal ideal domain it, follows that $\mathcal{O}_\beta$ is flat.

For a Λ-module X we will denote by X_ξ the following submodule $X_\xi = \{x \in X; (a-\xi)^l = 0 \quad \text{for some} \quad l\}$; it will be called *$\xi$-torsion* of X. (Recall that $\xi = \beta(0)$). Note that X_ξ is a finitely dimensional vector space if X is finitely generated over Λ.

8.1.3. Lemma. *Given a finitely generated Λ-module X, the map $X \to X \otimes_\Lambda \mathcal{O}_\beta$ which maps $x \in X$ to $x \otimes 1$, establishes an isomorpfism between the ξ-torsion submodule X_ξ and the $\mathcal{O}$-torsion submodule of $X \otimes_\Lambda \mathcal{O}_\beta$. Thus the $\mathcal{O}$-torsion submodule of $X \otimes_\Lambda \mathcal{O}_\beta$ contains a summand of the form $\mathcal{O}/t^l\mathcal{O}$ for every summand $\Lambda/(a-\xi)^l\Lambda$ in X_ξ.*

Proof. It is enough to prove the Lemma assuming that X is cyclic, i.e., X is free or isomorphic to $\Lambda/(a-\eta)^l\Lambda$ for some $\eta \in \mathbb{C}$ and $l >$

0. If X is free or $X = \Lambda/(a-\eta)^l\Lambda$ with $\eta \neq \xi$, then both torsion submodules in question vanish. Now we are left with the possibility $X = \Lambda/(a-\xi)^l\Lambda$; in this case one easily checks the statement of the lemma.

Let $\tilde{M}$ denote the covering of the manifold M corresponding to the kernel of $\phi : \pi_1(M) \to J$. The infinite cyclic group J acts on $\tilde{M}$ as the group of covering transformations. The homology $H_j(\tilde{M};\mathbb{C})$ are modules over Λ, called *Alexander modules.*

The torsion submodule of $H_j(\tilde{M};\mathbb{C})$ can be represented as a sum of cyclic modules

$$\Lambda/p_1\Lambda \oplus \Lambda/p_2\Lambda \oplus \cdots \oplus \Lambda/p_k\Lambda,$$

where $p_r \in \Lambda, \quad r = 1,2,\dots,k$ are non-zero Laurent polynomials determined uniquely up to units. The product $\Delta_j = p_1 \cdot p_2 \cdot \ldots \cdot p_k$ is called the *Alexander polynomial* in dimension j. It is a polynomial in a.

8.1.4. Proposition. *Suppose that a family of representations is given by an analytic curve $\beta : (-\epsilon,\epsilon) \to \mathbb{C}^*$ which passes through a point $\xi = \beta(0) \in \mathbb{C}^*$ such that the derivative $\beta'(0)$ is not zero. Then the dimension of the $\mathcal{O}$-torsion submodule τ^i of the $\mathcal{O}$-cohomology of the deformation is equal to the maximal power of the factor $(a-\xi)$ which divides the Alexander polynomial $\Delta_{i-1}(a)$ of dimension $i-1$. The $\mathcal{O}$-module τ^i is semi-simple if and only if the Λ-module $H_{i-1}(\tilde{M};\mathbb{C})_\xi$ is semi-simple.*

Proof. Using Theorem 7.4 and Lemma 8.2 we find that τ^i is isomorphic to the $\mathcal{O}$-torsion of $H^i(M;\mathcal{O}_\beta)$ which is given by

$$H^i(\mathrm{Hom}_\Lambda(C_*(\tilde{M});\mathcal{O}_\beta) = H^i(\mathrm{Hom}_\Lambda(\tilde{M};\Lambda)) \otimes_\Lambda \mathcal{O}_\beta = H^i(M;\Lambda) \otimes_\Lambda \mathcal{O}_\beta.$$

By Lemma 8.3 ξ-torsion $H^i(M;\Lambda)_\xi$ determines the $\mathcal{O}$-torsion in $H^i(M;\mathcal{O}_\beta)$. Now, the Kuneth formula

$$H^i(\mathrm{Hom}_\Lambda(\tilde{M};\Lambda)) = \mathrm{Hom}_\Lambda(H_i(\tilde{M});\Lambda) \oplus \mathrm{Ext}^1_\Lambda(H_{i-1}(\tilde{M});\Lambda)$$

can be used to express the ξ-torsion of $H^i(M;\Lambda)$ through the ξ-torsion of the Alexander module $H_{i-1}(\tilde{M};\mathbb{C})$. This completes the proof.

8.1.5. From Proposition 8.1.4 it follows that the Euler number χ of any deformation, determined by an analytic curve β on $\mathbb{C}^*$ as described above, equals the order at $a = \xi$ of the rational function

$$\prod_i \Delta_{2i-1}(a)/\Delta_{2i}(a),$$

where $\xi = \beta(0)$. This fact agrees with computation of Milnor [17] showing that the Reidemeister torsion (which is defined as a numerical function of the representation $a \in \mathbb{C}^*$ outside the set of roots of the Alexander polynomials of all dimensions) is given by the formula

$$\prod_i \Delta_{2i}(a)/\Delta_{2i-1}(a).$$

8.1.6. Now we will construct a precise example of a *flat line bundle with no semi-simple deformations.*

Fix two integers: $n \geq 3$ and i with $2 < 2i < n+1$. Consider the following $\mathbb{Z}[a, a^{-1}]$-module

$$A = \mathbb{Z}[a, a^{-1}]/(a^2 - a + 1)^2\mathbb{Z}[a, a^{-1}].$$

It is a module of type K in terminology of [15 (§3)]. Note that the complexification $A \otimes \mathbb{C}$ splits as

$$\Lambda/(a - \xi_+)^2\Lambda \oplus \Lambda/(a - \xi_-)^2\Lambda, \qquad \text{where} \quad \xi_\pm = \exp(\pm i\pi/3).$$

Using Theorem 9.1 of [15] we may construct an n-dimensional knot $k^n \subset S^{n+2}$, i.e., a smooth submanifold k^n with k^n diffeomorphic to the standard sphere S^n, such that the fundamental group of the complement $X = S^{n+2} - k^n$ is infinite cyclic (we will identify it with J) and the Alexander modules are the following:

$$H_j(\tilde{X}) \simeq \begin{cases} A, & \text{for } j = i \\ 0, & \text{for all } 1 < j < (n+1)/2, \quad j \neq i. \end{cases}$$

Here $\tilde{X}$ denotes the universal cover of X.

Now we want to perform a Dehn surgery along this knot. Let $k^n \times D^2$ be a tubular neighbourhood of k^n in S^{n+2}; cut it out and replace by $D^{n+1} \times S^1$ in the standard way. As the result we obtain a closed $(n+2)$-dimensional manifold M with infinite cyclic fundamental group, $\pi_1(M) \simeq J$, such that the Alexander modules are given by

$$H_j(\tilde{M}) \simeq \begin{cases} A, & \text{for } j = i \\ 0, & \text{for all } 1 < j < (n+1)/2, \quad j \neq i. \end{cases}$$

Thus we see that the ξ-torsion of the complexified Alexander module $H_i(\tilde{M}; \mathbb{C})_\xi$ is not semi-simple for $\xi = \xi_\pm$. By Proposition 8.1.4 we therefore conclude that the $\mathcal{O}$-torsion homology module τ^{i+1} will be not semi-simple for any curve passing through $\xi_\pm$ with nonzero velocity. Hence the Euler numbers χ of deformations corresponding to the above curves will be distinct (here we assume that the dimension n is odd) from the jump of the derived Euler characteristics, cf. §2.11.

Note additionally that these points $\xi_\pm$ lie on the unit circle and so we may find curves on the unit circle which pass through $\xi_\pm$; the latter would represent unitary families of representations having the above property.

8.2. Example 2: Koszul complexes. A.Dimca and M.Saito [7] studied deformations of Koszul complexes. It turned out that they behave entirely differently from deformations of elliptic complexes. Properties like Theorem 2.9, which state that the homology cannot increase immediately, are false for Koszul complexes.

Let $P = \mathbb{C}[x_1, x_2, \dots, x_n]$ be the ring of polynomials in $x_1, x_2, \dots, x_n$. The Koszul resolution

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots \xrightarrow{d} \Omega^n \to 0$$

is the complex (Ω, d) with Ω^k being the set of all *polynomial* k-forms in $x_1, x_2, \dots, x_n$ and with the boundary operator d being the exterior derivative. Thus $\Omega^0 = P$ and every Ω^k is a free P-module with the basis $dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_n}$, where $i_1 < i_2 < \dots < i_k$. Homology of this complex is nontrivial only in dimension 0, and the complex provides a free resolution of $\mathbb{C}$.

Let $f \in P$ be a fixed polynomial. Consider the Witten's deformation of the Koszul complex

$$D_f : \Omega^k \to \Omega^{k+1}, \; D_f(\omega) = d\omega + df \wedge \omega.$$

Note that Witten's gauge transformation

$$\omega \longmapsto e^f \omega$$

does not exist in this case because the form $e^f \omega$ *is not polynomial*. It turns out that homology of the complex (Ω, D_f) *depends* on f and can be *nontrivial*. It was explicitly computed by A.Dimca and M.Saito in [7]. They obtained the following answer.

Consider the map $f : \mathbb{C}^n \to \mathbb{C}$ defined by the polynomial f. Then f induces a topological fibration over a Zariski-open subset of $\mathbb{C}$. Let

$F = f^{-1}(z)$ denote the generic fiber, where $z \in \mathbb{C}$. Then the Theorem of A.Dimca and M.Saito [7] states that

$$H^{k+1}(\Omega, D_f) = \tilde{H}^k(F, \mathbb{C})$$

for any k.

From this Theorem it follows that the homology of the deformation

$$D_{tf}(\omega) = d\omega + tdf \wedge \omega$$

(where t is a parameter) does not depend on t for $t \neq 0$, $t \in \mathbb{C}$ since the polynomials f and tf have the same generic fiber.

Moreover, the deformed complex (Ω, D_{tf}) may have *bigger* homology than the original complex (Ω, d).

The simplest concrete example: let $n = 1$, $f = x^2$. The generic fiber consists of two points and so the complex (Ω, D_{tf}) has nontrivial 1-dimensional homology for any $t \neq 0$, while it is the 1-dimensional homology vanishes for $t = 0$.

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